Exercise 1

Solution.

1. τ_1 is a topology, since

$$\varnothing \subset \{a\} \subset \{a, b\} \subset X.$$

For any nonempty subfamily of this finite chain, the union is its largest member and the intersection is its smallest member. The empty union is $\varnothing$, and the empty intersection is X. The closed sets are the complements of the open sets:

$$\mathcal{F}_1 = \{\varnothing, X, \{b, c\}, \{c\}\}.$$

2. τ_2 is not a topology: it is not closed under unions,

$$\{a\}, \{b\} \in \tau_2, \qquad \{a\} \cup \{b\} = \{a, b\} \notin \tau_2.$$

3. τ_3 is not a topology: it is not closed under finite intersections,

$$\{a, b\}, \{b, c\} \in \tau_3, \qquad \{a, b\} \cap \{b, c\} = \{b\} \notin \tau_3.$$

4. τ_4 is not a topology: it is not closed under finite intersections,

$$\{a, c\}, \{b, c\} \in \tau_4, \qquad \{a, c\} \cap \{b, c\} = \{c\} \notin \tau_4.$$

Exercise 2

Solution.

Every topology on $X = \{p, q\}$ contains $\varnothing$ and X. We only need to decide which of the two singletons are open. This gives exactly four topologies:

Topology	Open sets	Closed sets
Indiscrete	$\{\varnothing, X\}$	$\{\varnothing, X\}$
τ_p	$\{\varnothing, \{p\}, X\}$	$\{\varnothing, \{q\}, X\}$
τ_q	$\{\varnothing, \{q\}, X\}$	$\{\varnothing, \{p\}, X\}$
Discrete	$\{\varnothing, \{p\}, \{q\}, X\}$	$\{\varnothing, \{p\}, \{q\}, X\}$

They satisfy the axioms: for the only two incomparable sets,

$$\{p\} \cup \{q\} = X, \qquad \{p\} \cap \{q\} = \varnothing.$$

All other nonempty unions and intersections return one of the chosen sets. The closed sets follow by taking complements.

Discrete topology: $\{p\}$ is both open and closed;

Indiscrete topology: $\{p\}$ is neither open nor closed,

since, in the second case, neither $\{p\}$ nor its complement $\{q\}$ is open.

Exercise 3

Solution.

1. By definition, $F \in \mathcal{F} \iff X \setminus F \in \tau$. Hence

$$X \setminus \varnothing = X \in \tau, \qquad X \setminus X = \varnothing \in \tau \quad \Longrightarrow \quad \varnothing, X \in \mathcal{F}.$$

If all the F_i are closed, De Morgan's laws give

$$X \setminus \bigcap_{i \in I} F_i = \bigcup_{i \in I} (X \setminus F_i) \in \tau \qquad \Longrightarrow \bigcap_{i \in I} F_i \in \mathcal{F},$$
$$X \setminus \bigcup_{i \in J} F_i = \bigcap_{i \in J} (X \setminus F_i) \in \tau \quad (J \text{ finite}) \quad \Longrightarrow \bigcup_{i \in J} F_i \in \mathcal{F}.$$

For an empty family, the intersection is X and the union is $\varnothing$.

2. Take the closed sets $F_n = [1/n, 1]$, $n \geqslant 1$. Then

$$\bigcup_{n \geqslant 1} F_n = (0, 1], \qquad \frac{1}{n} \in (0, 1], \quad \frac{1}{n} \longrightarrow 0 \notin (0, 1].$$

Thus the union is not closed. The equality follows because $0 < x \leqslant 1 \Rightarrow x \in F_n$ whenever $n \geqslant 1/x$.

Take the open sets $U_n = (-1/n, 1/n)$. We have

$$\bigcap_{n \geqslant 1} U_n = \{0\}, \qquad \forall \varepsilon > 0, \quad \varepsilon/2 \in (-\varepsilon, \varepsilon) \setminus \{0\}.$$

The intersection is not open. To justify the equality, if $x \neq 0$, choose an integer $n > 1/|x|$; then $x \notin U_n$.

Exercise 4

Solution.

1. Write $B(x, r) = (x - r, x + r)$. We have $\varnothing \in \tau_u$ vacuously, and $\mathbb{R} \in \tau_u$ since $B(x, 1) \subseteq \mathbb{R}$.

For a family $(U_i)_{i \in I} \subseteq \tau_u$,

$$x \in \bigcup_{i \in I} U_i \Longrightarrow \exists i_0 : \ x \in U_{i_0} \Longrightarrow \exists r > 0 : \ B(x, r) \subseteq U_{i_0} \subseteq \bigcup_{i \in I} U_i.$$

Thus the union is open. For $x \in U_1 \cap \cdots \cap U_m$, choose

$$B(x, r_k) \subseteq U_k \quad (r_k > 0), \qquad r = \min(r_1, \ldots, r_m) > 0.$$

Then $B(x, r) \subseteq \bigcap_{k=1}^{m} U_k$, so the finite intersection is open. The empty union is $\varnothing$, and the empty intersection is $\mathbb{R}$. Therefore τ_u is a topology.

2. The classification is

Set	$(0,1)$	$[0,1]$	$[0,1)$	$\{0\}$	$\mathbb{Q}$	$\varnothing$	$\mathbb{R}$
Open	yes	no	no	no	no	yes	yes
Closed	no	yes	no	yes	no	yes	yes

Indeed,

$$x \in (0, 1), \quad r = \tfrac{1}{2} \min(x, 1 - x) \Longrightarrow B(x, r) \subseteq (0, 1),$$
$$\mathbb{R} \setminus [0, 1] = (-\infty, 0) \cup (1, +\infty) \quad \text{is open},$$
$$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty) \quad \text{is open}.$$

The sets $[0, 1]$, $[0, 1)$ and $\{0\}$ are not open: they contain 0, but contain no interval $(-\varepsilon, \varepsilon)$, with $\varepsilon > 0$. Also,

$$\frac{1}{n+1} \in (0, 1), \quad \frac{1}{n+1} \longrightarrow 0 \notin (0, 1); \qquad 1 - \frac{1}{n} \in [0, 1), \quad 1 - \frac{1}{n} \longrightarrow 1 \notin [0, 1).$$

So $(0, 1)$ and $[0, 1)$ are not closed. Every nonempty open interval meets both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$; hence $\mathbb{Q}$ is neither open nor closed. Finally, $\varnothing$ and $\mathbb{R}$ are open and are each other's complements, so both are closed.

Exercise 5

Solution.

1. We have $\varnothing \in \tau_{cf}$ by definition, and $X \in \tau_{cf}$ since $X \setminus X = \varnothing$ is finite.
Let $U = \bigcup_{i \in I} U_i$, where $U_i \in \tau_{cf}$. If $U = \varnothing$, it is open. Otherwise, choose $U_{i_0} \neq \varnothing$. Then

$$X \setminus U \subseteq X \setminus U_{i_0} \quad \text{is finite.}$$

Hence $U \in \tau_{cf}$. For $V = \bigcap_{k=1}^{m} U_k$, if some U_k is empty, then $V = \varnothing$. Otherwise,

$$X \setminus V = \bigcup_{k=1}^{m} (X \setminus U_k) \quad \text{is finite, so } V \in \tau_{cf}.$$

The empty intersection is X. Thus τ_{cf} is a topology. Moreover,

$$F \text{ is closed} \iff X \setminus F \in \tau_{cf} \iff F = X \text{ or } F \text{ is finite.}$$

2. If X is finite, every $X \setminus U$ is finite. Therefore

$$\tau_{cf} = \mathcal{P}(X) \qquad \text{(the discrete topology).}$$

3. If X is infinite and $E \subseteq X$ is finite and nonempty, then

$$E \text{ open} \Longrightarrow X \setminus E \text{ finite} \Longrightarrow X = E \cup (X \setminus E) \text{ finite},$$

a contradiction. Thus no nonempty finite subset is open.

Exercise 6

Solution.

1. For $A = (-2, 0] \cup \{1\} \cup (2, 3)$, we obtain

$$\begin{aligned} \operatorname{Int} A &= (-2, 0) \cup (2, 3), \\ \overline{A} &= [-2, 0] \cup \{1\} \cup [2, 3], \\ \operatorname{Ext} A &= \mathbb{R} \setminus \overline{A} = (-\infty, -2) \cup (0, 1) \cup (1, 2) \cup (3, +\infty), \\ \partial A &= \overline{A} \setminus \operatorname{Int} A = \{-2, 0, 1, 2, 3\}. \end{aligned}$$

The intervals $(-2, 0)$ and $(2, 3)$ are open and contained in A. For $x = 0$ or 1, every $r > 0$ gives

$$\delta = \min(r/2, 1/4) > 0, \qquad x + \delta \in B(x, r) \setminus A.$$

Thus neither point is interior. For the closure, the set $[-2, 0] \cup \{1\} \cup [2, 3]$ is closed and contains A, and the added points are limits of the following sequences in A:

$$-2 + \frac{1}{n} \longrightarrow -2, \qquad 2 + \frac{1}{n+1} \longrightarrow 2, \qquad 3 - \frac{1}{n+1} \longrightarrow 3 \qquad (n \geqslant 1).$$

Finally,

$$0 \in A \setminus \operatorname{Int} A \Longrightarrow A \text{ is not open}, \qquad -2 \in \overline{A} \setminus A \Longrightarrow A \text{ is not closed.}$$

2. The given points are classified as follows:

Interior points	$-1,\ 5/2$
Exterior points	$3/2$
Boundary points	$-2,\ 0,\ 1,\ 2,\ 3$

In particular, 1 is a boundary point even though it is isolated in A: every interval centred at 1 meets both A and its complement.