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Topology — Problem Sheet 1 (Additional Exercises)

Ex1: (Convergence and uniqueness of limits)

i. Let (X, τ) be a Hausdorff topological space. Prove that every sequence in X has at most one limit.

ii. Show that the Hausdorff assumption cannot be omitted: construct a non-Hausdorff topological space and a sequence in that space that converges to two distinct points.

Ex2: (Continuity of the identity map)

Let τ_1 and τ_2 be topologies on the same set X. Characterise when the identity map

$$\mathrm{id} : (X, \tau_1) \longrightarrow (X, \tau_2), \qquad \mathrm{id}(x) = x,$$

is continuous.

Ex3: (A homeomorphism from $\mathbb{R}$ onto $(0, 1)$)

Construct a homeomorphism from $\mathbb{R}$ onto $(0, 1)$, with their usual topologies, and verify all required properties.

References

[1] University of Vienna. *Exercise sheet for Introduction to Topology, WS2022.*

[2] O. Ya. Viro, O. A. Ivanov, N. Yu. Netsvetaev, and V. M. Kharlamov. *Elementary Topology: Problem Textbook.* American Mathematical Society, 2008. Section 9, Exercise 9.1.

[3] Anton Petrunin. *A Minimalist Intro to Topology.* Introductory lecture notes, Penn State. Section 5A, Exercise 5.3.

Solutions

Exercise 1: Recall.

- A sequence $(x_n)_{n\geq 1}$ *converges* to $x \in X$, written $x_n \to x$, if for every open neighbourhood U of x there is an integer $N \geq 1$ such that $x_n \in U$ for every $n \geq N$. In words, the sequence is eventually in every open neighbourhood of its limit.
- A space is *Hausdorff* if any two distinct points have disjoint open neighbourhoods: for $x \neq y$, there exist $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$.

Solution to (i). Suppose that the same sequence (x_n) converges to both x and y. We prove that $x = y$. If $x \neq y$, the Hausdorff property gives disjoint open neighbourhoods U of x and V of y.

Since $x_n \to x$, there is $N_1 \geq 1$ such that

$$n \geq N_1 \quad \Longrightarrow \quad x_n \in U.$$

Similarly, since $x_n \to y$, there is $N_2 \geq 1$ such that

$$n \geq N_2 \quad \Longrightarrow \quad x_n \in V.$$

For every $n \geq \max\{N_1, N_2\}$, we would therefore have $x_n \in U \cap V = \varnothing$, which is impossible. Thus $x = y$, and every sequence has at most one limit. □

Solution to (ii). Let $X = \{a, b\}$, where $a \neq b$, and equip it with the *indiscrete topology*

$$\tau = \{\varnothing, X\}.$$

This is a topology: it contains $\varnothing$ and X, and any union or finite intersection of its members is again either $\varnothing$ or X. The space is not Hausdorff, because the only open neighbourhood of either a or b is X, so disjoint open neighbourhoods cannot exist.

Consider the constant sequence $x_n = a$ for every $n \geq 1$. For either point $p \in \{a, b\}$, every open neighbourhood of p is X, and $x_n \in X$ for all $n \geq 1$. The definition of convergence is therefore satisfied with $N = 1$ for both points. Hence

$$x_n \to a \quad \text{and} \quad x_n \to b, \qquad a \neq b.$$

In fact, every sequence in an indiscrete space converges to every point of that space. □

Exercise 2: Recall. A map $f : (X, \tau_X) \to (Y, \tau_Y)$ is *continuous* if $f^{-1}(V) \in \tau_X$ for every $V \in \tau_Y$, where $f^{-1}(V) = \{x \in X : f(x) \in V\}$ denotes the preimage of V. A topology τ_1 is *finer* than τ_2 if $\tau_2 \subseteq \tau_1$.

Solution. For every subset $U \subseteq X$, the identity map satisfies $\mathrm{id}^{-1}(U) = U$. Consequently,

$$\begin{aligned} \mathrm{id} : (X, \tau_1) \to (X, \tau_2) \text{ is continuous} &\iff \mathrm{id}^{-1}(U) \in \tau_1 \text{ for every } U \in \tau_2 \\ &\iff U \in \tau_1 \text{ for every } U \in \tau_2 \\ &\iff \tau_2 \subseteq \tau_1. \end{aligned}$$

Thus the required characterisation is

$$\mathrm{id} : (X, \tau_1) \longrightarrow (X, \tau_2) \text{ is continuous} \quad \Longleftrightarrow \quad \tau_2 \subseteq \tau_1.$$

Exercise 3: Recall. The *usual topology* on $\mathbb{R}$ consists of unions of open intervals. The *subspace topology* on $A \subseteq \mathbb{R}$ consists of the sets $A \cap U$, where U is open in $\mathbb{R}$; this gives the usual topology on $(0, 1)$. A *homeomorphism* is a bijective continuous map whose inverse is also continuous.

Construction. Define

$$f : \mathbb{R} \longrightarrow (0, 1), \qquad f(x) = \frac{e^x}{1 + e^x}.$$

(1) **Well-definedness and continuity.** Since $e^x > 0$ and $e^x < 1 + e^x$, we have $0 < f(x) < 1$ for every $x \in \mathbb{R}$. The exponential function is continuous and the denominator $1 + e^x$ never vanishes, so f is continuous as a real-valued function, hence also as a map into the subspace $(0, 1)$.

(2) **Injectivity.** If $f(x) = f(z)$, cross-multiplication gives

$$e^x(1+e^z) = e^z(1+e^x) \quad \Longrightarrow \quad e^x = e^z.$$

Since the exponential function is injective, $x = z$.

(3) **Surjectivity and inverse.** Given $y \in (0,1)$, the number $y/(1-y)$ is positive, so we may set $x = \log\big(y/(1-y)\big)$. Then $e^x = y/(1-y)$ and

$$f(x) = \frac{y/(1-y)}{1+y/(1-y)} = y.$$

Thus f is surjective, and its inverse is

$$f^{-1}(y) = \log\left(\frac{y}{1-y}\right), \qquad 0 < y < 1.$$

(4) **Continuity of the inverse.** The function $y \mapsto y/(1-y)$ is continuous and positive on $(0,1)$, and the natural logarithm is continuous on $(0,\infty)$. Their composition f^{-1} is therefore continuous.

Hence, f is a homeomorphism from $\mathbb{R}$ onto $(0,1)$. □