Continuity and closure

Proposition. Let X and Y be topological spaces and let $f\colon X \to Y$. Then f is continuous if and only if

$$f(\overline{A}) \subseteq \overline{f(A)} \quad \text{for every } A \subseteq X.$$

Here $\overline{A}$ denotes closure in X, and $\overline{f(A)}$ denotes closure in Y. Also, $f^{-1}(C) = \{x \in X : f(x) \in C\}$ is the inverse image of C; no inverse function is required.

Two facts to remember. The closure $\overline{A}$ is the smallest closed set containing A. Thus, if a closed set F contains A, then it contains $\overline{A}$. Also, f is continuous exactly when inverse images of closed sets are closed. This follows from the definition using open sets by taking complements, since

$$f^{-1}(Y \setminus C) = X \setminus f^{-1}(C).$$

Proof. ($\Rightarrow$) Suppose f is continuous. Fix any $A \subseteq X$ and set

$$F = f^{-1}\left(\overline{f(A)}\right).$$

The set $\overline{f(A)}$ is closed, so continuity makes F closed in X. Moreover, if $a \in A$, then $f(a) \in f(A) \subseteq \overline{f(A)}$; hence $a \in F$. Thus $A \subseteq F$.

Because F is closed and contains A, it also contains $\overline{A}$. Consequently, every $x \in \overline{A}$ satisfies $x \in F$, which means $f(x) \in \overline{f(A)}$. Therefore

$$f(\overline{A}) \subseteq \overline{f(A)}.$$

Since A was arbitrary, this holds for every subset of X.

($\Leftarrow$) Suppose the inclusion holds for every $A \subseteq X$. Let $C \subseteq Y$ be any closed set and put $A = f^{-1}(C)$. We will show that A is closed.

By the definition of A, we have $f(A) \subseteq C$. Since C is closed, it contains the closure of $f(A)$, so $\overline{f(A)} \subseteq C$. Applying the assumed inclusion to this A gives

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq C.$$

Thus, if $x \in \overline{A}$, then $f(x) \in C$, so $x \in f^{-1}(C) = A$. Hence $\overline{A} \subseteq A$. But always $A \subseteq \overline{A}$, so $A = \overline{A}$, and A is closed.

Therefore the inverse image of every closed set is closed. This proves that f is continuous.□

Separable spaces

Countable set. Here, *countable* means finite or countably infinite. A countably infinite set can be listed as $a_1, a_2, a_3, \ldots$, with every element appearing in the list. Examples $\mathbb{N}$, $\mathbb{Z}$.

Dense subset. A subset $A \subseteq X$ is *dense in* X if $\overline{A} = X$, this is equivalent to

$$U \cap A \neq \varnothing \quad \text{for every nonempty open set } U \subseteq X.$$

Indeed, if $\overline{A} = X$, choose $x \in U$ and use the closure criterion. Conversely, if every nonempty open set meets A, then every open set containing any $x \in X$ meets A; hence every x belongs to $\overline{A}$.

In a metric space (X, d), density can be written as

$$\text{for every } x \in X \text{ and every } \varepsilon > 0, \quad \text{there is } a \in A \text{ with } d(x, a) < \varepsilon.$$

Here $d(x, a)$ is the distance between x and a. This works because every open neighborhood of x contains an open ball centered at x. Thus, points of A can approximate every point of X as closely as desired.

Defintion (separable space). A topological space is *separable* if it has a countable dense subset:

$$X \text{ is separable} \quad \Longleftrightarrow \quad \exists A \subseteq X : \ A \text{ is countable and } \overline{A} = X.$$

Only the dense subset A must be countable; the whole space X can be uncountable. The main example is $\mathbb{R}$, with the countable dense subset $\mathbb{Q}$ and the usual topology.

Example 1. $\mathbb{R}$ with its usual topology is separable. Take $A = \mathbb{Q}$. We check the two required properties (Countability and Density). Exercise.

Example 2. Every countable topological space is separable. Take $A = X$. This set is countable, and $\overline{X} = X$. For instance, $\mathbb{N}$ and $\mathbb{Q}$, with any topology, are separable.

Connection with the Proposition: continuous images. If X is separable and $f\colon X \to Y$ is continuous and onto ($f(X) = Y$), then Y is separable.

Proof. Choose a countable dense subset $A \subseteq X$. Its image $f(A)$ is countable: applying f to a list of the elements of A gives a list of the image, possibly with repetitions. Since $\overline{A} = X$, the Proposition gives

$$Y = f(X) = f(\overline{A}) \subseteq \overline{f(A)} \subseteq Y.$$

Hence $\overline{f(A)} = Y$. Thus $f(A)$ is a countable dense subset of Y, which proves that Y is separable. □