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Topology — Problem Sheet 2

Instructions. Justify every answer. For a false assertion, give an explicit counterexample.

Notation. For a metric d, write

$$B_d(a,r) = \{x : d(a,x) < r\}, \qquad C_d(a,r) = \{x : d(a,x) \le r\} \quad (r > 0).$$

The notation $\overline{A}$ denotes the closure of A.

Exercise 1.

For $x, y \in \mathbb{R}$, consider

$$d_1(x,y) = 2|x-y|, \quad d_2(x,y) = |x-y|^2, \quad d_3(x,y) = \sqrt{|x-y|}, \quad d_4(x,y) = |x^2 - y^2|.$$

Determine which of these functions are metrics on $\mathbb{R}$.

Exercise 2.

Let (X,d) be a nonempty metric space, and set

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)} \qquad (x,y \in X).$$

(a) Show that $\varphi(t) = t/(1+t)$ is increasing on $[0,\infty)$ and that $\varphi(s+t) \le \varphi(s)+\varphi(t)$ for $s,t \ge 0$. Deduce that ρ is a metric.

(b) For $a \in X$ and $0 < r < 1$, prove that $B_\rho(a,r) = B_d(a, r/(1-r))$. Determine $B_\rho(a,r)$ when $r \ge 1$.

(c) Prove that (X,ρ) is bounded. For $X = \mathbb{R}$ and $d(x,y) = |x-y|$, calculate $\operatorname{diam}_\rho(\mathbb{R})$ and decide whether this diameter is attained by a pair of points.

Exercise 3.

Let X be a nonempty set, and define $\delta(x,y) = 0$ if $x = y$ and $\delta(x,y) = 1$ otherwise.

(a) Prove that δ is a metric.

(b) Determine $B_\delta(a,r)$ and $C_\delta(a,r)$ for each of the cases $0 < r < 1$, $r = 1$ and $r > 1$.

(c) Show that every subset of X is both open and closed. Determine the diameter of every nonempty subset of X.

Exercise 4.

Let $\mathcal{C} = C([0,1],\mathbb{R})$ be the set of continuous real-valued functions on $[0,1]$. For $f, g \in \mathcal{C}$, set

$$d_\infty(f,g) = \sup_{t \in [0,1]} |f(t) - g(t)|.$$

(a) Explain why this supremum is finite and is attained. Prove that d_∞ is a metric on $\mathcal{C}$.

(b) Calculate $d_\infty(f,g)$ for $f(t) = t$ and $g(t) = t^2$.

(c) Describe $B_{d_\infty}(0,1)$ and $C_{d_\infty}(0,1)$, where 0 denotes the zero function. Decide whether $t \mapsto t$ and $t \mapsto t/2$ belong to each ball.

(d) With $f(t) = t$, $u_n(t) = t + 1/n$ and $v_n(t) = t^n$, calculate $d_\infty(u_n, f)$ and $d_\infty(v_n, 0)$. Which of these distances tends to zero?

Exercise 5.

Let (X, d) be a metric space.

(a) Prove the reverse triangle inequality $|d(x, a) - d(y, a)| \leq d(x, y)$ for all $x, y, a \in X$.

(b) Prove directly that every open ball is open, every closed ball is closed, and every singleton is closed.

(c) Using the definition in terms of balls, prove that the open subsets of X form a topology: $\varnothing$ and X are open, arbitrary unions of open sets are open, and finite intersections of open sets are open.

(d) Show that $\overline{B_d(a, r)} \subseteq C_d(a, r)$. Using a discrete metric space with at least two points, show that equality need not hold.

Exercise 6.

Let Y be a nonempty subset of a metric space (X, d), with the restricted distance $d_Y = d|_{Y \times Y}$.

(a) Prove that d_Y is a metric and that, for $a \in Y$ and $r > 0$,

$$B_Y(a, r) = Y \cap B_X(a, r), \qquad C_Y(a, r) = Y \cap C_X(a, r).$$

(b) Prove that $U \subseteq Y$ is open in Y if and only if $U = Y \cap V$ for some open set V in X. Deduce the corresponding characterisation of closed sets in Y.

(c) Take $X = \mathbb{R}$ and $Y = [0, 2)$. Compute $B_Y(0, 1)$ and $C_Y(0, 1)$. Classify each of

$$Y, \qquad [0, 1), \qquad (0, 1], \qquad [1, 2), \qquad \{0\}$$

as open only, closed only, both, or neither in Y.

(d) Give a subset that is open in Y but not open in $\mathbb{R}$, and a subset that is closed in Y but not closed in $\mathbb{R}$.

Exercise 7.

Recall that $x_n \to x$ in (X, d) means that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$.

(a) Prove that a convergent sequence has a unique limit and that every subsequence converges to the same limit.

(b) If $x_n \to x$ and $y_n \to y$, prove that $d(x_n, y_n) \to d(x, y)$. Start by establishing

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y).$$

(c) For $A \subseteq X$ and $x \in X$, prove that $x \in \overline{A}$ if and only if there is a sequence (a_n) in A converging to x.

(d) Deduce that A is closed if and only if every sequence in A that converges in X has its limit in A.

Exercise 8.

Let (X, d) be a metric space.

(a) Prove that every convergent sequence is Cauchy.

(b) Prove that every Cauchy sequence is bounded, meaning that its set of terms is bounded. Is every bounded sequence Cauchy? Give a counterexample in $\mathbb{R}$.

(c) Prove that every subsequence of a Cauchy sequence is Cauchy.

(d) Suppose that a Cauchy sequence (x_n) has a subsequence converging to some $x \in X$. Prove that the whole sequence converges to x.