

Definition: let X be a set, and let

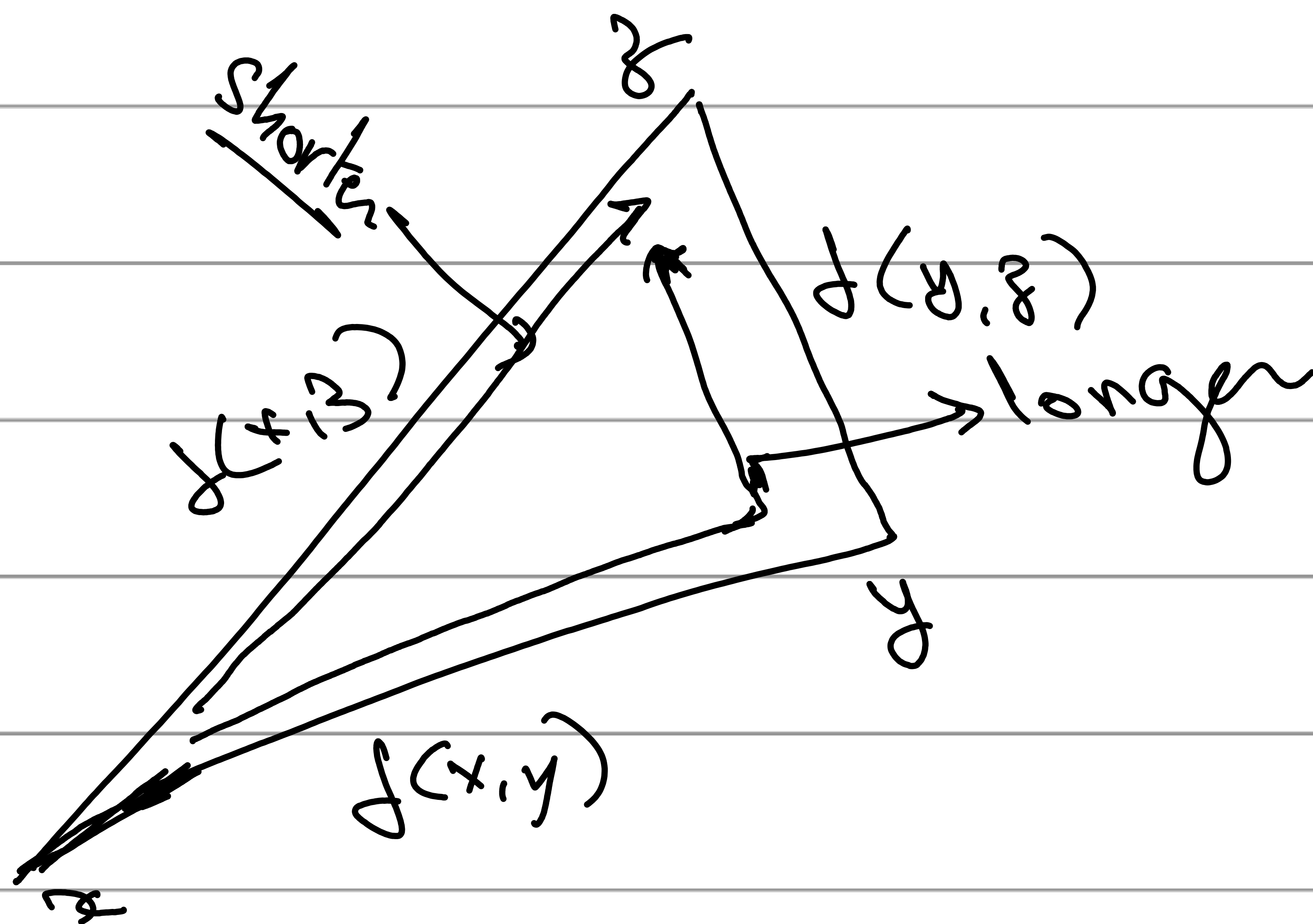
$d: X \times X \rightarrow \mathbb{R}$ be a function

such that for all $x, y, z \in X$:

1. $d(x, y) \geq 0$. (nonnegativity)
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$ (symmetry)
4. $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality).

The pair (X, d) is called a metric space

The function d is called the metric or the distance function.



* Diagram of the triangle inequality in metric spaces

Example:

$\mathbb{R}$ is a metric space with the metric

$$d(x, y) := |x - y|.$$

$$\begin{aligned} * d(x, z) &= |x - z| = |x - y + y - z| \leq |x - y| + |y - z| \\ &= d(x, y) + d(y, z). \end{aligned}$$

* this metric is a standard metric on $\mathbb{R}$

Example: We can also put a different metric

on $\mathbb{R}$. For example:

$$d(x, y) := \frac{|x - y|}{|x - y| + 1}.$$

* Note that $d(x, y) = \varphi(|x - y|)$ where

$\varphi(t) = \frac{t}{t+1}$ and φ is an increasing function (positive derivative).

Hence:

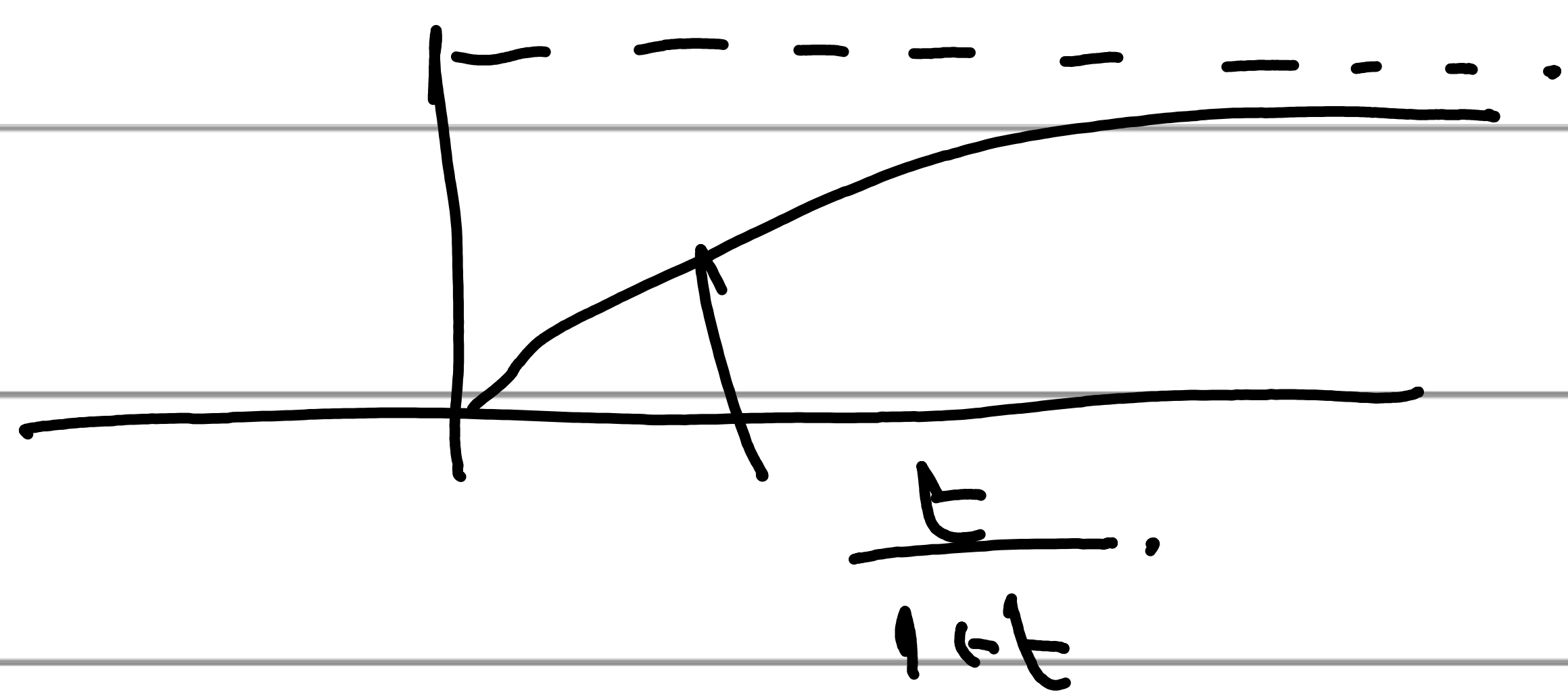
$$d(x, z) = \varphi(|x - z|)$$

$$= \varphi(|x - y + y - z|)$$

$$\leq \varphi(|x - y| + |y - z|)$$

$$= \frac{|x - y| + |y - z|}{|x - y| + |y - z| + 1} = \frac{|x - y|}{|x - y| + |y - z| + 1} + \frac{|y - z|}{|x - y| + |y - z| + 1}$$

$$\leq \frac{|x - y|}{|x - y| + 1} + \frac{|y - z|}{|y - z| + 1} = d(x, y) + d(y, z).$$



Example:

For any set X . define $d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$

Example: let $C([a, b], \mathbb{R})$ be a set of continuous real-valued functions on the interval $[a, b]$. Define the metric on $C([a, b], \mathbb{R})$ as

$$d(f, g) := \sup_{x \in [a, b]} |f(x) - g(x)|$$

(TD).

Proposition: let (X, d) be a metric space and $Y \subset X$. then the restriction $d|_{Y \times Y}$ is metric on Y .

Definition: If (X, d) is a metric space, $Y \subset X$, and $d' := d|_{Y \times Y}$, then (Y, d') is said to be a subspace of (X, d) .

* Definition: let (X, d) be metric space.

A subset $S \subset X$ is said to be bounded if there exists a $p \in X$ and $\alpha \in \mathbb{R}$:

$$d(p, x) \leq \alpha \quad \forall x \in S.$$

We say (X, d) is bounded if X itself is a bounded subset

* Remark:

Suppose $X \neq \emptyset$. Then $S \subset X$ being bounded is equivalent to either.

1. $\forall p \in X, \exists \alpha > 0 : d(x, p) \leq \alpha \quad \forall x \in S$.
 2. $\text{diam}(S) := \sup \{ d(x, y) : x, y \in S \} < \infty$.
- $\text{diam}(S)$ is called the diameter of a set.

* Definition: let (X, d) M.S., and $\varepsilon > 0$. Define the open ball, or simply ball, of radius ε around x as

$$B(x, \varepsilon) := \{ y \in X : d(x, y) < \varepsilon \}$$

Define the closed ball as

$$C(x, \varepsilon) := \{ y \in X : d(x, y) \leq \varepsilon \}.$$

Note: when dealing with different metric spaces, it is sometimes vital to emphasize which metric space the Ball is in.

We do this by writing

$$B_x(x, \varepsilon) := B(x, \varepsilon)$$

$$C_x(x, \varepsilon) := C(x, \varepsilon).$$

Example: Take the metric space $\mathbb{R}$ with the standard metric. For $x \in \mathbb{R}$

$$\text{and } \varepsilon > 0, B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$$

$$C(x, \varepsilon) = [x - \varepsilon, x + \varepsilon].$$

Example: Be careful when working on a subspace. Consider the metric space

$[0, 1]$ as a subspace of $\mathbb{R}$. Then in $[0, 1]$

$$B(0, 1/2) = B_{[0, 1]}(0, 1/2)$$

$$= \{y \in [0, 1] : |0 - y| < 1/2\}$$

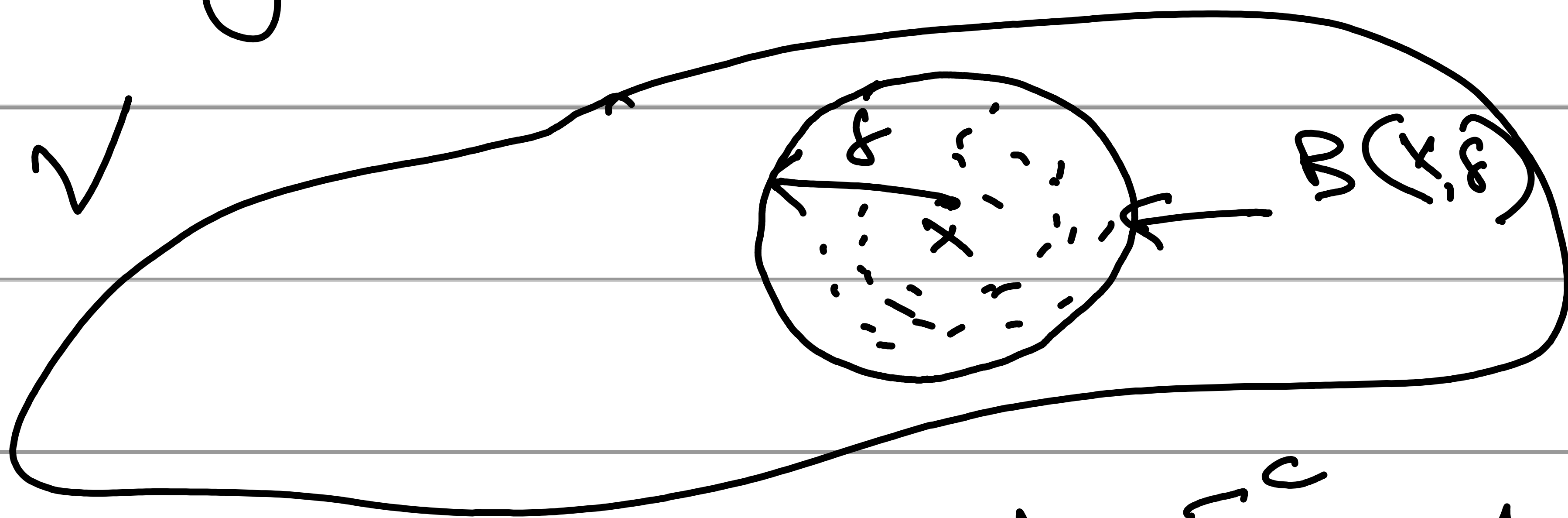
$$= [0, 1/2).$$

This is different from $B_{\mathbb{R}}(0, 1/2) = (-1/2, 1/2)$.

* Definition:

Let (X, d) M.S. A subset $V \subset X$ is open

if for every $x \in V$ $\exists \varepsilon > 0 : B(x, \varepsilon) \subset V$



* A subset $E \subset X$ is closed if $E^c = X \setminus E$ is open

* if $x \in V$ and V is open, then we say V is open neighborhood.

Example:

1. The set $(0, \infty) \subset \mathbb{R}$ is open. $\forall x \in (0, \infty)$

let $\delta := x$ then $B(x, \delta) = (0, 2x) \subset (0, \infty)$.

2. The set $[0, \infty) \subset \mathbb{R}$ is closed: Given

$x \in (-\infty, 0) = [0, \infty)^c$, let $\delta := -x$. Then

$B(x, \delta) = (-2x, 0) \subset (-\infty, 0) = [0, \infty)^c$.

3. If (X, d) is any metric space, and $x \in X$ is a point, then $\{x\}$ is closed (exercise).

On the other hand, $\{x\}$ may or may not be open depending on X . The set $\{0\} \subset \mathbb{R}$ is not open as $B(0, \delta)$ contains nonzero numbers for every $\delta > 0$.

if $X = \{x\}$, then $\{x\}$ is open.

Proposition: let (X, d) be a metric space.

1. $\emptyset$ and X are open.
2. If $V_1, V_2, \dots, V_k$ are open subset of X
Then $\bigcap_{j=1}^k V_j$ is also open.

3. If $\{V_\lambda\}_{\lambda \in I}$ is an arbitrary collection of open subsets of X , then $\bigcup_{\lambda \in I} V_\lambda$ is also open.

Proof:

1. $\emptyset, X$ are obviously open in X .
2. If $x \in \bigcap_{j=1}^k V_j \Rightarrow x \in V_j$ for all j .
As V_j are all open. $\forall j \exists \delta_j > 0$
such that $B(x, \delta_j) \subset V_j$.

Take $\delta_j = \min \{\delta_1, \delta_2, \dots, \delta_k\}$ and notice $\delta > 0$
We have

$$B(x, \delta) \subset B(x, \delta_j) \subset V_j \quad \forall j$$

and so, $B(x, \delta) \subset \bigcap_{j=1}^k V_j$.

Consequently the intersection is
open.

3. If $x \in \bigcup_{\lambda \in I} V_\lambda \Rightarrow x \in V_\lambda$ for some $\lambda \in I$.

As V_λ open; $\exists \delta > 0 : B(x, \delta) \subset V_\lambda$.

But then $B(x, \delta) \subset \bigcup_{\lambda \in I} V_\lambda$, and so
the union is open.

Proposition

Let (X, d) be a metric space
 $x \in X$, and $\delta > 0$. Then $B(x, \delta)$ is open
and $\overline{B(x, \delta)}$ is closed.

Proof: let $y \in B(x, \delta)$.

Set $\alpha := \delta - d(x, y)$. As $\alpha > 0$,

Consider $z \in B(y, \alpha)$. Then.

$$d(x, z) \leq d(x, y) + d(y, z)$$

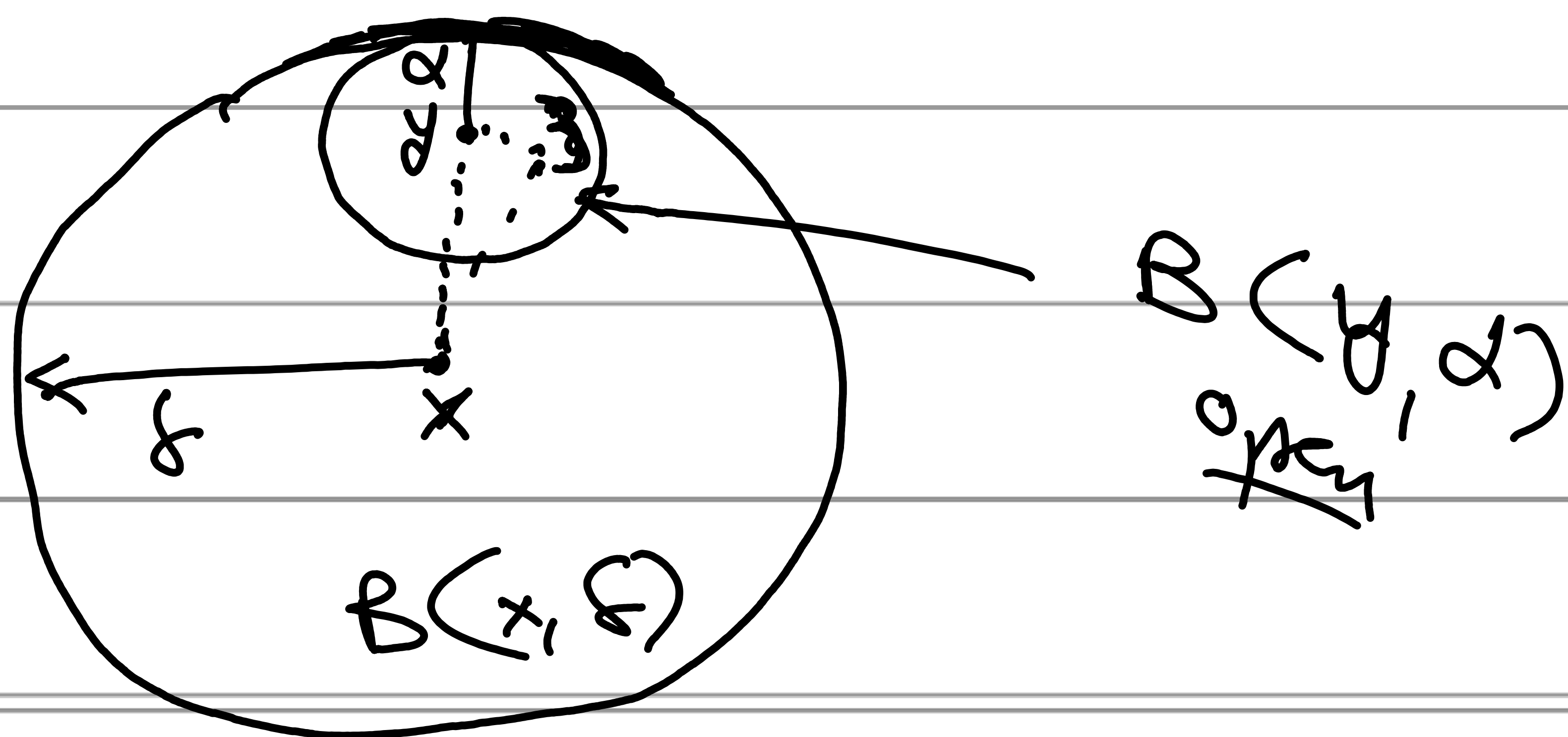
$$< d(x, y) + \alpha$$

$$= d(x, y) + \delta - d(x, y) = \delta.$$

Therefore, $z \in B(x, \delta) \quad \forall z \in B(y, \alpha)$.

So $B(y, \alpha) \subset B(x, \delta)$, and so $B(x, \delta)$
is open.

$B(x, \delta)$ is open.



Proposition: (X, d) is a metric space, and
 $Y \subset X$. Then

$U \subset Y$ is open in $Y \iff \exists$ open set $V \subset X$:
 $V \cap Y = U$.

Proof: Suppose $V \subset X$ is open and $V \cap Y = U$.

Let $x \in U$, as V open and $x \in V$, $\exists \delta > 0$
such that $B_x(x, \delta) \subset V$. Then.

$$B_Y(x, \delta) = B_x(x, \delta) \cap Y \subset V \cap Y = U.$$

So U is open in Y .

The proof of the opposite direction, that is,
that if $U \subset Y$ is open in the subspace
topology there exists a V (rste exercise)
