

Base for a Topology:

Let $(X, \mathcal{T})$ be a T.S. A basis for the Topology $\mathcal{T}$ is a collection $\beta \subseteq \mathcal{T}$ such that every open set in X is a union of elements of β .

For every open set $U \in \mathcal{T}$ and every point $x \in U$, $\exists B \in \beta : x \in B \subseteq U$.

Example:

1. The open intervals (a, b) form basis for the usual topology on $\mathbb{R}$.

2. $X \neq \emptyset$; $\mathcal{T} = \mathcal{P}(X)$; $\beta = \{ \{x\} : x \in X \}$

Then β is always a base for discrete topology.

(Since every subset $U \subseteq X$ can be written as $U = \bigcup_{x \in U} \{x\}$)

3. $X = \{1, 2, 3\}$

$\mathcal{T} = \{ \emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\} \}$

$\beta = \{ \{1\}, \{2\}, \{3\} \}$

(Since every open set is a union of

these singletons, for example $\{1, 3\} = \{1\} \cup \{3\}$

The empty set is the empty union.

Theorem: let $(X, \mathcal{C})$ be a topo-space. A collection $\mathcal{B} = \{B_\alpha : \alpha \in I\}$ of sets in $\mathcal{C}$ is a base for $\mathcal{C} \iff$ for any open set U and any point $x \in U$, there is B_α such that $x \in B_\alpha \subseteq U$.

Definition

let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be topo-spaces.

A sequence (x_n) in X converges to $x \in X$ if for every open set U containing x there exists $N \in \mathbb{N} : n \geq N \implies x_n \in U$.

We write $x_n \rightarrow x$.

Example

In $\mathbb{R}$ with the usual topo. the sequence

$$x_n = \frac{1}{n} \quad ; \quad n \geq 1 \quad \text{converges to } 0.$$

Indeed, every open set U containing 0

contains an interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

take $N > \frac{1}{\varepsilon}$. Then, for every $n \geq N$.

$$0 < \frac{1}{n} < \varepsilon,$$

So $x_n \in U$. Thus $x_n \rightarrow 0$.

Definition:

Let (X, τ_x) and (Y, τ_y) be a topo. spaces.
A map $f: X \rightarrow Y$ is continuous if
The inverse image of every open set in Y
is open in X :

$$V \in \tau_y \Rightarrow f^{-1}(V) \in \tau_x.$$

Example:

Equip $\mathbb{R}$ with its usual topo. The map.

$$f: \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto f(x) = 2x.$$

is continuous.

Proof: Let $V \subseteq \mathbb{R}$ be open and let

$x \in f^{-1}(V)$. Since $2x \in V$, there

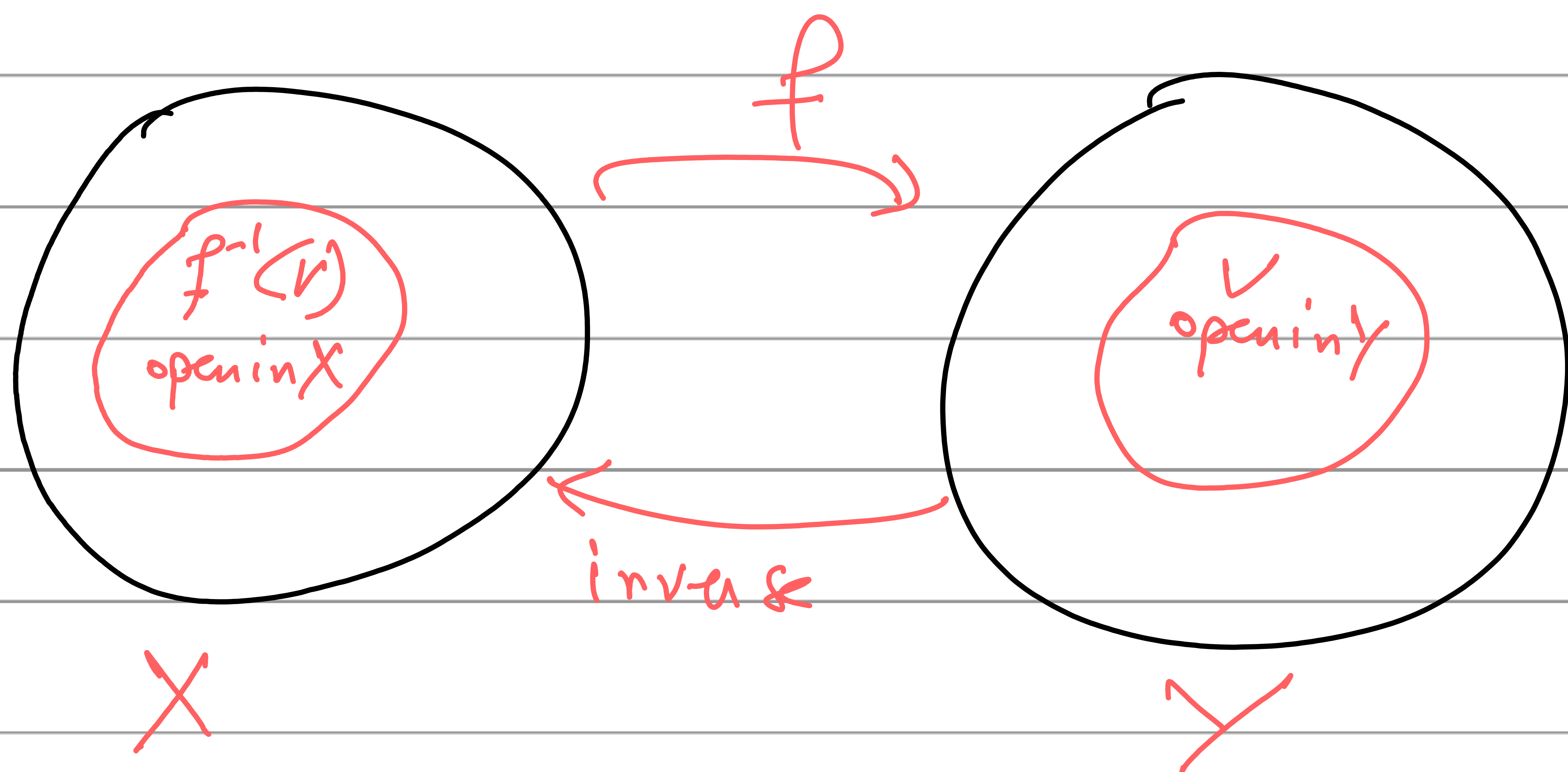
exists $\varepsilon > 0$ such that $(2x - \varepsilon, 2x + \varepsilon) \subseteq V$.

Consequently:

$$\left(x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}\right) \subseteq f^{-1}(V).$$

Thus $f^{-1}(V)$ is open. Therefore
 f is continuous 

* Continuous map:



The map f is continuous when every open set $V \subseteq Y$ has an open inverse image $f^{-1}(V) \subseteq X$.

Th:

Let X and Y be topological spaces. A map $f: X \rightarrow Y$ is continuous $\Leftrightarrow$ the inverse image of every closed set in Y is closed in X .

Proof:

$\Rightarrow$ Suppose that f is continuous. Let $F \subseteq Y$ be closed.

Since F is closed $\Rightarrow C_Y^F$ is open in Y

By continuity $f^{-1}(C_Y^F)$ is open in X .

But

$$f^{-1}(C_Y^F) = C_X^{f^{-1}(F)}$$

Thus the complement of $f^{-1}(F)$ is open in X , so $f^{-1}(F)$ is closed in X .

$\boxed{\Leftarrow}$ Suppose that $f^{-1}(F)$ is closed in X for every closed set $F \subseteq Y$.

Let $V \subseteq Y$ be open. Then C_Y^V is closed in Y . By our assumption;

$f^{-1}(C_Y^V)$ is closed in X .

Using

$$f^{-1}(C_Y^V) = C_X^{f^{-1}(V)}$$

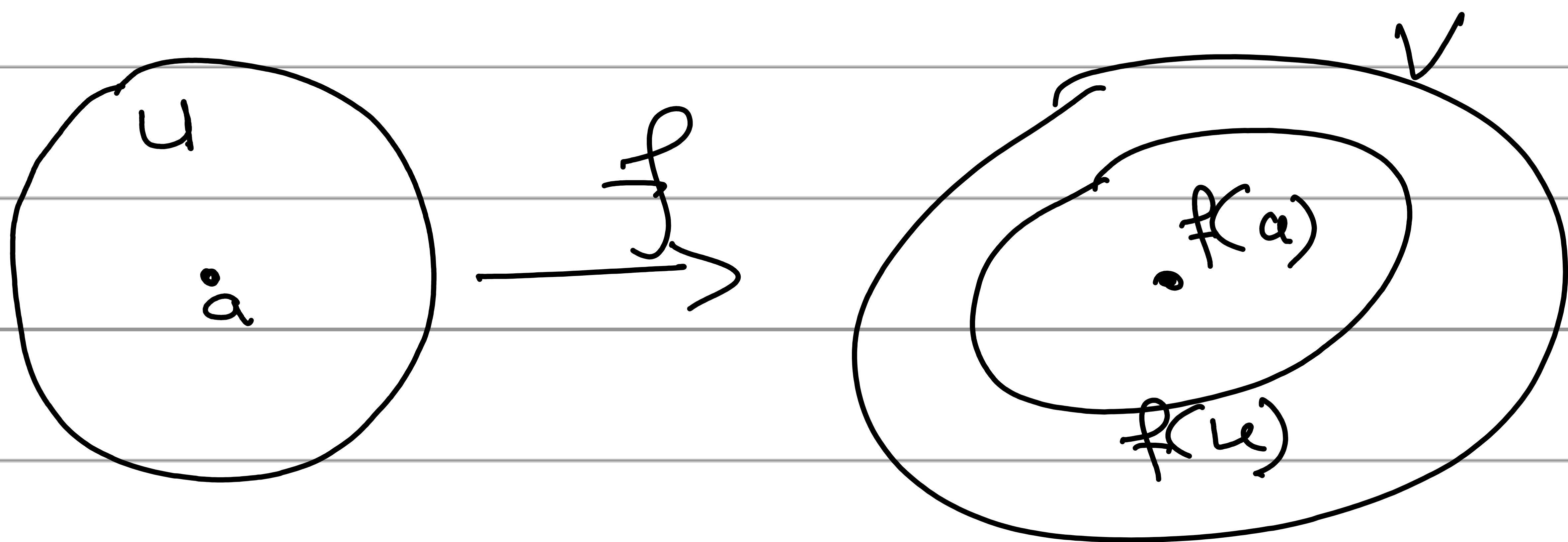
Hence the complement of $f^{-1}(V)$ is closed in X , so $f^{-1}(V)$ is open in X .

Which means that f is continuous



Definition (Continuity at a point)

Let X and Y be top-spaces, and
let $a \in X$. A map $f: X \rightarrow Y$ is
continuous at a if, for every open set
 $V \subseteq Y$ containing $f(a)$, there exists
an open set $U \subseteq X$ containing a such
that $f(U) \subseteq V$.



for every open neighborhood V of $f(a)$,
there exists an open neighborhood U
of a such that

$$f(U) \subseteq V.$$



Example:

let $X = \{a, b, c\}$ and $Y = \{1, 2\}$

$$\tau_X = \{ \emptyset, \{a\}, \{a, b\}, X \}$$

$$\tau_Y = \{ \emptyset, \{1\}, \{2\}, Y \}$$

Define $f: X \rightarrow Y$ by

$$f(a) = f(b) = 1$$

$$f(c) = 2.$$

Then f is continuous at a and b
but not at c .

* at a : choose $U = \{a\}$. For every open neighborhood V of 1 , we have $f(U) = \{1\} \subseteq V$.

* at b : choose $U = \{a, b\}$, again, $f(U) = \{1\} \subseteq V$.

* at c : take $V = \{2\}$. The only open neighborhood of c is X and
 $f(X) = \{1, 2\} \not\subseteq V$

Remark:

A function $f: X \rightarrow Y$ is continuous on X if f is continuous at every point of X .

Th: let X, Y, Z be topo-spaces. and
 $f: X \rightarrow Y$, $g: Y \rightarrow Z$
be continuous functions. then
 $g \circ f: X \rightarrow Z$ is continuous.

Proof: (exercise).

Definition (Homeomorphism)

let X and Y be topo-spaces. A map $f: X \rightarrow Y$ is a homeomorphism if:

1. f is bijective

2. f is continuous

3. its inverse $f^{-1}: Y \rightarrow X$ is continuous.

(if such map exists, X and Y called homeomorphic)
 $X \cong Y$.

Example:

$$1. f: \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto f(x) = 2x+1$$

is a homeomorphism:

1. It is bijective.

2. Continuous.

3. $f^{-1}(y) = \frac{y-1}{2}$ is also continuous.

Since it is an affine function.

$$2. f: (a,b) \xrightarrow{\text{open}} (c,d) \xleftarrow{\text{open}}$$

$$a < b.$$

$$c < d.$$

$$x \mapsto f(x) = c + \frac{d-c}{b-a} (x-a)$$

(exercise).

Remark:

i) $I: X \rightarrow X$ is a homeomorphism i.e. $X \cong X$.

ii) If $f: X \rightarrow Y$ is homeo, then f^{-1} is homeo.

iii) If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are homeo.
then $g \circ f: X \rightarrow Z$ is homeo.

~~QED~~