

Topology — Problem Sheet 1

Ex 1: Let $X = \{a, b, c\}$. Determine whether each family below is a topology on X :

$$\begin{aligned}\tau_1 &= \{\emptyset, X, \{a\}, \{a, b\}\}, & \tau_2 &= \{\emptyset, X, \{a\}, \{b\}\}, \\ \tau_3 &= \{\emptyset, X, \{a, b\}, \{b, c\}\}, & \tau_4 &= \mathcal{P}(X) \setminus \{\{c\}\}.\end{aligned}$$

If it is, list its closed sets. If not, identify a failed axiom and give an explicit counterexample.

Ex 2: Let $X = \{p, q\}$. Find all topologies on X and list the open and closed sets for each. Identify the discrete and indiscrete topologies. Give an example of a set that is both open and closed, and one that is neither open nor closed, specifying the topology in each case.

Ex 3: 1. Let (X, τ) be a topological space and $\mathcal{F}$ its family of closed sets. Prove that $\emptyset, X \in \mathcal{F}$ and, for every family $(F_i)_{i \in I} \subseteq \mathcal{F}$,

$$\bigcap_{i \in I} F_i \in \mathcal{F}, \quad \bigcup_{i \in J} F_i \in \mathcal{F} \quad \text{for every finite } J \subseteq I.$$

2. In $\mathbb{R}$ with its usual topology, construct sequences of closed sets $(F_n)_{n \geq 1}$ and open sets $(U_n)_{n \geq 1}$ such that

$$\bigcup_{n=1}^{\infty} F_n \text{ is not closed,} \quad \bigcap_{n=1}^{\infty} U_n \text{ is not open.}$$

Ex 4: Let

$$\tau_u = \{U \subseteq \mathbb{R} : \forall x \in U, \exists \varepsilon > 0,]x - \varepsilon, x + \varepsilon[\subseteq U\}.$$

1. Prove that τ_u is a topology on $\mathbb{R}$.
2. Classify each set below as open only, closed only, both, or neither in $(\mathbb{R}, \tau_u)$. Justify your answers.

$$(0, 1), \quad [0, 1], \quad [0, 1), \quad \{0\}, \quad \mathbb{Q}, \quad \emptyset, \quad \mathbb{R}.$$

Ex 5: Let X be a set and define

$$\tau_{\text{cf}} = \{\emptyset\} \cup \{U \subseteq X : X \setminus U \text{ is finite}\}.$$

1. Prove that τ_{cf} is a topology on X and determine its closed sets.
2. Identify τ_{cf} when X is finite.
3. If X is infinite, can a nonempty finite subset of X be open? Justify your answer.

Ex 6: In $\mathbb{R}$ with its usual topology, let

$$A = (-2, 0] \cup \{1\} \cup (2, 3).$$

1. Determine $\text{Int } A$, $\overline{A}$, $\text{Ext } A$, and ∂A . Is A open, closed, both, or neither?
2. Classify the points

$$-2, \quad -1, \quad 0, \quad 1, \quad \frac{3}{2}, \quad 2, \quad \frac{5}{2}, \quad 3$$

as interior, exterior, or boundary points of A .

Ex 7: Let (X, τ) be a topological space and $A \subseteq X$.

1. Using the pointwise definition of interior, prove that

$$\text{Int } A = \bigcup \{U \in \tau : U \subseteq A\}.$$

2. Using the neighbourhood definition of closure, prove that

$$\text{cl } A = \bigcap \{F \subseteq X : A \subseteq F, F \text{ is closed}\}.$$

3. Deduce that interior and closure are

$$\text{Int}(\text{Int } A) = \text{Int } A, \quad \text{cl}(\text{cl } A) = \text{cl } A,$$

and for all $A \subseteq B \subseteq X$,

$$\text{Int } A \subseteq \text{Int } B, \quad \text{cl } A \subseteq \text{cl } B.$$

Ex 8: Let X be a topological space and $A \subseteq X$.

1. Prove that

$$\text{Ext } A = \text{Int}(X \setminus A) = X \setminus \text{cl } A,$$

$$\text{Int } A = X \setminus \text{cl}(X \setminus A),$$

$$\partial A = \text{cl } A \cap \text{cl}(X \setminus A) = \text{cl } A \setminus \text{Int } A.$$

2. Deduce that ∂A is closed.

Ex 9: Let X be a topological space and $A \subseteq X$. Prove the equivalences

$$A \text{ is closed} \iff \partial A \subseteq A,$$

$$A \text{ is open} \iff A \cap \partial A = \emptyset,$$

$$A \text{ is clopen} \iff \partial A = \emptyset.$$

Ex 10: Let X be a topological space and $A, B \subseteq X$.

1. Prove that

$$\text{Int}(A \cap B) = \text{Int } A \cap \text{Int } B, \quad \text{cl}(A \cup B) = \text{cl } A \cup \text{cl } B.$$

2. Prove the inclusions

$$\text{Int } A \cup \text{Int } B \subseteq \text{Int}(A \cup B), \quad \text{cl}(A \cap B) \subseteq \text{cl } A \cap \text{cl } B.$$

For each, give an example in $\mathbb{R}$ with its usual topology where the inclusion is strict.

Ex 11*: Let (X, τ) be a topological space. For $A \subseteq X$, denote by A' its derived set (set of accumulation points of A) and by $\text{Iso}(A)$ its set of isolated points.

1. Prove that

$$\text{cl } A = A \cup A', \quad A \text{ is closed} \iff A' \subseteq A, \quad \text{Iso}(A) = A \setminus A'.$$

2. In $\mathbb{R}$ with its usual topology, let

$$S = \left\{ \frac{1}{n} : n \geq 1 \right\}, \quad T = S \cup \{0\}, \quad B = ((-1, 1) \setminus \{0\}) \cup \{2\}.$$

For each $A \in \{S, T, B\}$, determine

$$\text{Int } A, \quad \text{cl } A, \quad \text{Ext } A, \quad \partial A, \quad A', \quad \text{Iso}(A),$$

and decide whether A is open or closed.

Ex 12:** Let

$$X = \{a, b, c\}, \quad \tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \quad A = \{b\}, \quad B = \{a\}.$$

1. Determine the smallest open neighbourhood N_x of each $x \in X$.
2. Using these neighbourhoods, compute and justify, for each $E \in \{A, B\}$,

$$\text{Int } E, \quad \text{cl } E, \quad \text{Ext } E, \quad \partial E, \quad E', \quad \text{Iso}(E),$$

where $\text{Iso}(E)$ denotes the set of isolated points of E .

3. Determine all $x \in X$ for which $\{x\}$ is closed.