* **Definition:**

Let X be non-empty set.

A collection of sub sets of X is said to be the topology τ on X if the following properties are satisfied:

1. $\phi, X \in \tau$
2. If $U_i \in \tau$ for all $i \in \Gamma$, then $\bigcup_{i \in \Gamma} U_i \in \tau$
3. If $U_1, \dots, U_n \in \tau$, then $\bigcap_{k=1}^{n} U_k \in \tau$.

i.e: Arbitrary union of open sets is open.
finite intersection of open sets is open.

* **Note**

- the members of τ are open sets and their Complements are closed sets.
- The pair (X, τ) or X is said to be Topological space

* **Examples**

$X = \{1, 2, 3, 4\}$.

$\tau_1 = \{\phi, X, \{1\}\}$ ✓

$\tau_2 = \{\phi, X, \{2\}\}$ ✓

$\tau_3 = \{\phi, X, \{1\}, \{2\}\}$ ✗.

$\tau_4 = \{\phi, \{1\}, \{3\}, \{1,2,3,4\}\}$ ✗.

$\tau_8 = \{\phi, X, \{1\}, \{4\}, \{1,4\}\}$ ✓.

* Discrete Topology:

Let $X \neq \phi$; $\tau = \mathcal{P}(X)$

Then τ is called discrette topology and X with this Topo is called discrete space

. Remark:

If $\tau_1 \subseteq \tau_2$ then

we call τ_2 is stronger (finer) then τ_1

Then discrete topology is the stronget Topology.

* Indiscrete topo: (Trivial topo).

Let $X \neq \phi$. $\tau = \{\phi, X\}$.

Then τ is called indiscrete (trivial) Topo.

. Remark:

The Trivial Topo is the weakest topo.

* Usual Topology:

A collection of subsets of $\mathbb{R}$ which can be expressed as a union of open intervals, forms a topology on $\mathbb{R}$ and is called usual Topology on $\mathbb{R}$.

A set $U \subseteq \mathbb{R}$ is open exactly when $\forall x \in U, \exists \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq U$.

* Cofinite topology:

Let X be an infinite set.

$\tau = \{ \phi, A_\alpha : A_\alpha \subseteq X ; A_\alpha^c \text{ is finite} \}$

then τ is a topology on X, known as cofinite topology.

example:

$\tau = \{\phi\} \cup \{ U \subseteq \mathbb{R} : \mathbb{R} \setminus U \text{ is finite} \}$

is a topology.

. Remark:

In cofinite topological space, open sets are infinite and closed sets are finite.

. Examples:

1. $X = \{1,2,3\}$; $\tau_1 = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$

Open sets : $\phi, X, \{1\}, \{1,2\}, \{1,3\}$.

Closed sets: $\phi, X, \{2,3\}, \{3\}, \{2\}$.

2. $X = \{1,2,3\}$; $\tau_2 = \{\phi, X, \{2\}, \{2,3\}\}$

Open sets: $\phi, X, \{2\}, \{2,3\}$.

Closed sets: $X, \phi, \{1,3\}, \{1\}$.

So;

$\{3\}, \{1,2\}$ are neither open nor closed.

. Theorem: Let (X, τ) be a topological space.

Then

(i) ϕ, X are closed.

(ii) Union of finite no. of closed sets of τ is closed.

(iii) Arbitrary intersection of closed sets of $\mathcal{C}$ is closed.

. Proof:

(i) let $\phi \in \mathcal{C}$ So ϕ is open

$\Rightarrow$ ϕ^c is closed

$\equiv$ X is closed.

let $X \in \mathcal{C}$ So X open.

$\Rightarrow X^c$ is closed $\equiv \phi$ is closed.

(ii) let $\{O_i : i = 1, 2, \ldots, n\}$ be finite Collection of open sets of $\mathcal{C}$. then

$\bigcap_{i=1}^{n} O_i \in \mathcal{C}$ is open

$\Rightarrow \left(\bigcap_{i=1}^{n} O_i\right)^c$ is closed Then by Morgan's law.

$\Rightarrow \bigcup_{i=1}^{n} O_i^c$ is closed.

Hence finite union of closed sets of $\mathcal{C}$ is closed.

(iii) Let $\{O_i, i \in I\}$ be an arbitrary Collection of members of $\mathcal{C}$. Then

$\bigcup_{i \in I} O_i \in \mathcal{C}$ is open.

$\Rightarrow \left(\bigcup_{i \in I} O_i\right)^c$ is closed. . (Morgan's)

$\Rightarrow \bigcap_{i \in I} O_i^c$ is closed. ∎

Note:

1. ϕ, X are open and closed.
2. Arbitrary union of open sets of $\mathcal{C}$ is open.
3. Finite intersection of open sets of $\mathcal{C}$ is open.
4. // union of closed sets of $\mathcal{C}$ is closed.
5. Arbitrary union of closed sets of $\mathcal{C}$ is closed.

In $\mathbb{R}$. usuelle topo. on $\mathbb{R}$, every $\left]-\frac{1}{n}, \frac{1}{n}\right[$ are open., but

$$\bigcap_{n \geq 1} \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\},$$

not open.

* Closure of a set:

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then.

The Closure of a subset $A \subseteq X$, denoted $\bar{A}$ or $cl(A)$, is the smallest closed set containing A:

$$\bar{A} = \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}.$$

equivalently:

$x \in \bar{A} \iff$ every open neighborhood of x intersects A.

in $\mathbb{R}$. the closure of $(0,1)$ is $[0,1]$.

* Interior point of a set:

Let $(X, \mathcal{T})$ be a topo space. and $A \subseteq X$.

$x \in A$ is an interior point of A if $\exists$ an open set U such that

$$x \in U \subseteq A.$$

The set of all interior points of A is called the int of A, denoted $Int(A)$ or A°.

. Note: interior of A is the largest open set contained in A. i.e. $A^\circ \subseteq A$.

. Example:

$$X = \{1,2,3,4,5\}$$

$$\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$A = \{3,4\}$$

$$A^\circ = \{3\}$$

Closed set:

i) A is closed $\Leftrightarrow$ A^c is open.

ii) A is closed $\Leftrightarrow$ every limit point of A belongs to A.

(iii) A is closed $\Leftrightarrow$ $\bar{A} = A$

iv) A is closed $\Leftrightarrow$ $Fr(A) \subseteq A$ (i.e) every frontier point of A belongs to A.

. Open set:

1. A is open if $A \in \tau$.
2. A " " $\Leftrightarrow$ $A^\circ = A$.
3. If every interior point of A belongs to A. Then A is open.

. Example:

$X = \{1, 2, 3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$

$A = \{1, 3\}$; $B = \{2\}$; $C = \{2, 3\}$, $D = \{3\}$

$A^\circ = \{1\}$

$B^\circ = \{2\} = C^\circ$; $D^\circ = \phi$.

* Limit point

Let (X, τ) be a topological space and $A \subseteq X$.

A point $x \in X$ is called limit point of A if every open set containing x has non-empty intersection with $A - \{x\}$.

limit point is also called cluster point, derived point, accumulation point.

Un point x est un point d'accumulation de A si chaque ouvert contenant x contient aussi un point de A différent de x.

$x \in A' \Leftrightarrow (\forall U \in \tau, x \in U \Rightarrow U \cap (A \setminus \{x\}) \neq \emptyset)$.

. Example:

$$X = \{1, 2, 3, 4, 5\}$$

$$\tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

$$A = \{3, 4\}$$

1. Is 1 Limit point of A?

all open sets containing 1

$$X, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}$$

So:

$\{3,4\} \cap \{1\} = \emptyset$. then, 1 is not.

2. for 2?

Open: $X, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}$.

$\{3,4\} \cap \{2\} = \emptyset$; then 2 is not

3. for 3?

$$X, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}$$

$A - \{3\} \cap \{3\} = \emptyset$.

4. for 4?

Open: X.

In $\mathcal{T}$. The unique open "contenant" 4 is X.

We have $3 \in X$., and 3 in A. and $3 \neq 4$.

So Yes.

~~6~~. for 5.?

X open.!

3 and 3 in X and 3, 4 in A.

$3, 4 \neq 5$.

Yes.

The set of limit points is

$A' = \{4, 5\}$.

Theorem: Let $(X, \mathcal{T})$ T.S. and $A \subseteq X$.

Then;

(i) $A^\circ \subseteq A$.

(ii) $(A^\circ)^\circ = A^\circ$

(iii) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$.

$A^\circ = A - Fr(A)$

Proof:

Suppose $A \subseteq B \Rightarrow A^\circ \subseteq A \subseteq B$.

So A° is open subset of B but B° is largest open subset of B.

$\Rightarrow A^\circ \subseteq A \subseteq B^\circ \subseteq B$

$\Rightarrow A^\circ \subseteq B^\circ$.

(iv) $(A \cap B)^\circ = A^\circ \cap B^\circ$. (exercise).

(v) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

* Exterior point of a set:

Let A be a subset of a topological space X.
A point $x \in X$ is an exterior point of A if there exists an open set U such that

$$x \in U \subseteq X \setminus A$$

Thus, x is an interior point of the complement of A.

i.e. x is exterior point of A if it is an interior point of A^c. $\left(Ext(A) = Int(A^c)\right)$

* The exterior of A is

$$Ext(A) = Int(X \setminus A) = X \setminus \overline{A}$$

. Example: $X = \{1,2,3,4,5\}$

$$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$$

$$A = \{2,3,4\} \Rightarrow A^c = \{1,5\}.$$

$$\Rightarrow Ext(A) = \{1\}.$$

. Theorem

Let $(X,2)$. T.S. and $A \subseteq X$. $B \subseteq X$.

. Then.

1. $(A^\circ)^c = \overline{A^c}$
2. $A^\circ = A - \overline{A^c}$
3. $Ext(X) = \phi$; $Ext(\phi) = X$.
4. $Ext(A) = (\bar{A})^c$
5. $Ext(A \cup B) = Ext(A) \cap Ext(B)$.
6. $Ext(A \cap B) \supseteq Ext(A) \cup Ext(B)$.

. Proof:

(exercice TP).

* Boundary (Frontier) point:

$(X,2)$. T.S; $A \subseteq X$.

a point $x \in X$ is called boundary (Frontier) point of A if every open set containing x has non-empty intersection with A and A^c.

and the set of all boundary (frontier) points of A is called boundary (frontier) of A. : $Fr(A)$ or $Fr(A) = \bar{A} \cap (A^c)$.

. Example

$$X = \{1,2,3,4\}$$

$$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$$

$A = \{1,2,3\}$; $A^c = \{4\}$

$$Fr(A) = \{3,4\}.$$

i.e.

$A = \{1,2,3\}$; $X \setminus A = \{4\}$

$Fr(A) = \bar{A} \setminus Int(A)$

$Int(A) = \{1,2\}$. ; $\bar{A} = X$.

So;

$$Fr(A) = \{3,4\}.$$

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* Theorem:

1. $Fr(A) = Fr(A^c)$; 2. $A^\circ = A \setminus Fr(A)$

3. $\bar{A} = A \cup Fr(A)$

4. $Fr(A)$ is closed subset of X.

5. A open and closed $\Leftrightarrow$ $Fr(A) = \phi$.

6. $Fr(A) \subseteq A \Leftrightarrow A$ is closed.

Proof: T.D.

Corollary:

1. A is closed $\Leftrightarrow Fr(A) \subseteq A$.
   i.e A is closed $\Leftrightarrow$ every frontier point of A belongs to A.
2. $Fr(A)$ is closed subset of X.
3. A is both open and closed $\Leftrightarrow Fr(A) = \emptyset$.

Proof.

(T.D)

* Neighbour of a point:

let $(X, \mathcal{C})$ be a top.s. and $x \in X$.

A subset $N \subseteq X$ is a neighborhood of x if there exists an open set $U \in \mathcal{C}$ such that

$$x \in U \subseteq N$$

In other words, N is a neighborhood of x is an interior point of N.

*Remark:

i. if N is open, then it is called open neighbourhood.

ii) If N is closed, it is called closed neighbourhood.

* Neighborhood system:

$(X, \mathcal{T})$. T.S and $x \in X$. The neighborhood system of x, denoted $\mathcal{N}(x)$, is a collection of all neighborhood of x.

$$\mathcal{N}(x) = \{N \subseteq X : \exists U \in \mathcal{T}, x \in U \subseteq N\}.$$

* Sub space:

let $(X, \mathcal{T}_X)$ be a. T. S. and $Y \subseteq X$.

The subspace topology on Y is

$$\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}_X\}.$$

The pair $(Y, \mathcal{T}_Y)$ is called a. Topo. sub-space of X.

. Remark.

A set is open in Y if it is the intersection of Y with an open set of X.

. Examples:

* if $X = \{1,2,3,4\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

and $Y = \{1,3\}$, then $\tau_Y = \{\phi, \{1\}, \{1,3\}\}$.

* $\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,2,3\}, \{1,3\}, \{2,3\}\}$

$Y = \{2,3,4\}$

$\tau_Y = \{\phi, Y, \{2\}, \{3\}, \{2,3\}\}$.

. Theorem:

(X, τ_X) be a T. space and (Y, τ_Y) be its subspace. Then, every open subset of (Y, τ_Y) is open in $(X, \tau_X) \Leftrightarrow$ Y is open in (X, τ_X).

. Proof:

(T.D.)