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1 The Complex Number System

Mathematics is the most beautiful and most powerful creation of the human spirit.
– Stefan Banach (1892-1945, Polish mathematician)

1.1 Complex Numbers

Let x and y be real numbers.

- The imaginary unit i is a number such that

$$\boxed{i = +\sqrt{-1}, \quad i^2 = -1, \quad i \notin \mathbb{R}}$$

- The rectangular form of a complex number is an expression of the form

$$\boxed{z = x + iy, \text{ where } x, y \in \mathbb{R}}$$

- The real and imaginary parts of $z = x + iy$ are the real numbers

$$\boxed{\Re(z) = x, \quad \operatorname{Im}(z) = y.}$$

- The conjugate $\overline{z}$ of the complex number $z = x + iy$ is

$$\boxed{\overline{z} = x - iy}$$

- The set of all complex numbers is denoted

$$\boxed{\mathbb{C} = \{z = x + iy : \ x, y \in \mathbb{R} \text{ and } i^2 = -1\}}$$

Arithmetic Operations.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are complex numbers, then the arithmetic operations of addition, subtraction, multiplication and division can be carried out as follows:

$$\begin{aligned}
z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) := (x_1 + x_2) + i(y_1 + y_2) \\
z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) := (x_1 - x_2) + i(y_1 - y_2) \\
z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) := (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\
\frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} := \frac{x_1 + iy_1}{x_2 + iy_2}\frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1x_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}; (z_2 \neq 0)
\end{aligned}$$

Laws of Complex Arithmetic

Let $z_1, z_2, z_3 \in \mathbb{C}$, then we have the following algebraic properties:

1. **Closure:** $z_1 + z_2 \in \mathbb{C}, \quad z_1 + z_2 \in \mathbb{C}$
2. **Additive and multiplicative identity:** $z + 0 = z$ and $1 \cdot z = z$, for all $z \in \mathbb{C}$
3. **Commutative laws:** $z_1 + z_2 = z_2 + z_1$ and $z_1 \cdot z_2 = z_2 \cdot z_1$
4. **Associate laws:** $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$
5. **Distributive laws:** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$
6. **Inverses:** $z_1 + (-z_1) = 0$ and $z_1 \cdot \frac{1}{z_1} = 1, (z_1 \neq 0)$
7. **Zero factors:** $z_1 \cdot z_2 = 0 \Rightarrow z_1 = 0$ or $z_2 = 0$

The above laws make $(\mathbb{C}, +, \cdot)$ into a **field** (Corps) .

This is the set of numbers obtained by appending i to the real numbers.

$$\mathbb{C} = \mathbb{R} + i\mathbb{R}$$

So the real numbers can be viewed as a subset of $\mathbb{C}$ because $\mathbb{R} = \mathbb{R} + i0 \subset \mathbb{C}$.
The same can be said for the pure imaginary numbers $i\mathbb{R} = 0 + i\mathbb{R} \subset \mathbb{C}$.

Algebraic Construction of Complex Numbers

If we endow $\mathbb{R}^2$ with the following operation:

$$\begin{array}{ll}
\text{Equality} & [a, b] = [c, d] \Longleftrightarrow a = c, b = d \\
\text{Addition} & [a, b] + [c, d] = [a + c, b + d] \\
\text{Multiplication} & [a, b] \cdot [c, d] = [ac - bd, bc + ad]
\end{array}$$

One can easily show that the above operations of addition and multiplication are commutative, associative and that multiplication is distributive with respect to addition. Then topologically speaking we say that $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic.

Numbers of the form $[a, 0]$ behave like real numbers so we identify $a := [a, 0]$. We also identify $i := [0, 1]$ and the pure imaginary numbers $ib := [0, 1][b, 0]$. Hence for any complex number $z = a + ib$ we have

$$z = a + ib = [a, 0] + [0, 1][b, 0] = [a, 0] + [0, b] = [a, b].$$

The real numbers correspond to the x-axis in the Euclidean plane. The complex numbers of the form iy are called purely imaginary numbers. They form the **imaginary axis** $i\mathbb{R}$ in the complex plane, which corresponds to the y-axis in the Euclidean plane.

We know for real algebra that the equation $x^2 + 1 = 0$ does not have a solutions in $\mathbb{R}$ since $x^2 + 1 > 0$. However one can see that

$$i^2 = (i)(i) = [0, 1][0, 1] = [(0)(0) - (1)(1), (0)(1) + (1)(0)] = [-1, 0] := -1$$

Hence the above equation identifies $i^2 = -1$ and shows that $\pm i$ are solutions to the equation $x^2 + 1 = 0$.

Remark.

It is remarkable that the addition of i lets us not only solve the equation $x^2 + 1 = 0$, but every polynomial equation. In fact if $z \in \mathbb{C}$ then

$$p(z) = a_n z^n + \cdots + a_1 z + a_0$$

is complex polynomial of degree $n > 0$, where $a_0, \ldots a_n$ are complex numbers, and $a_n \neq 0$. A key property of the complex numbers, not enjoyed by the real numbers, is that any polynomial with complex coefficients can be factored as a product of linear factors.

Fundamental Theorem of Algebra. Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization

$$p(z) = c(z - z_1)^{m_1} \cdots (z - z_k)^{m_k},$$

where the z_j's are distinct and $m_j \geq 1$. This factorization is unique, up to a permutation of the factors.

Example. The polynomial $p(x) = x^2 + 1$ with real coefficients cannot be factored as a product of linear polynomials with real coefficients, since it does not have any real roots. However, the complex polynomial $p(z) = z^2 + 1$ has the factorization

$$z^2 + 1 = (z - i)(z + i),$$

corresponding to the two complex roots $\pm i$ of $z^2 + 1$.

Solving Equations

Example: Solve $z^2 + 2z + 1 = 0$.

$$z^2 + 2z + 1 = 0 \Rightarrow z = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i.$$

Example: Solve $z^2 = \overline{z}$.

$$\begin{aligned} z^2 = \overline{z} &\Rightarrow (x^2 - y^2) + i(2xy) = x - iy \\ &\Rightarrow x^2 - y^2 = x \text{ and } 2xy = -y \\ &\Rightarrow x = -\frac{1}{2} \text{ and } y^2 = \frac{3}{4} \text{ or } y = 0 \text{ and } x = 0, 1. \end{aligned}$$

The solutions are given by

$$z = 0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

Absolute value and complex conjugate

For x and y real and $z = x + iy$ we define:

- $\overline{z} = x - iy$ as the **conjugate** of the complex number z.
- $|z| = \sqrt{x^2 + y^2}$ as the **absolute value** or **modulus** of the complex number z.

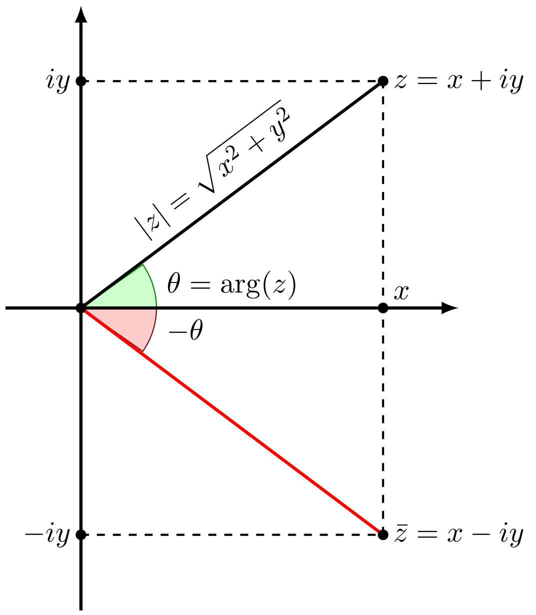


Figure 1.1: Absolute value and complex conjugate.

Properties. Let $z, w \in \mathbb{C}$ then we have:

(1) $\Re z = (z + \overline{z})/2$

(2) $\operatorname{Im} z = (z - \overline{z})/(2i)$

(3) $|z|^2 = z\overline{z} = x^2 + y^2$

(4) $\overline{(\overline{z})} = z$

(5) $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}, (z \neq 0)$

(6) $1/\overline{z} = z/|z|^2, (z \neq 0)$
(7) $z = \bar{z} \Leftrightarrow \operatorname{Im} z = 0$
(8) $z = -\bar{z} \Leftrightarrow \Re z = 0$
(9) $\overline{z \pm w} = \bar{z} \pm \bar{w}$
(10) $\overline{zw} = \bar{z}\bar{w}$
(11) $\overline{z^n} = (\bar{z})^n$
(12) $\overline{z/w} = \bar{z}/\bar{w}, (w \neq 0)$
(13) $\Re z \leq |z|$ and $\operatorname{Im} z \leq |z|$
(14) $|z| = |\bar{z}|$
(15) $|zw| = |z||w|$
(16) $\left|\frac{z}{w}\right|| = \frac{|z|}{|w|} =, (w \neq 0)$
(17) $|z \pm w| \leq |z| + |w|$
(18) $|z \pm w| \geq ||z| - |w||$

Addition and subtraction

Addition and subtraction of complex numbers are identical to addition and subtraction of real numbers. Thus,

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

The Argand representations of two complex numbers and their sum are shown in the figures below.

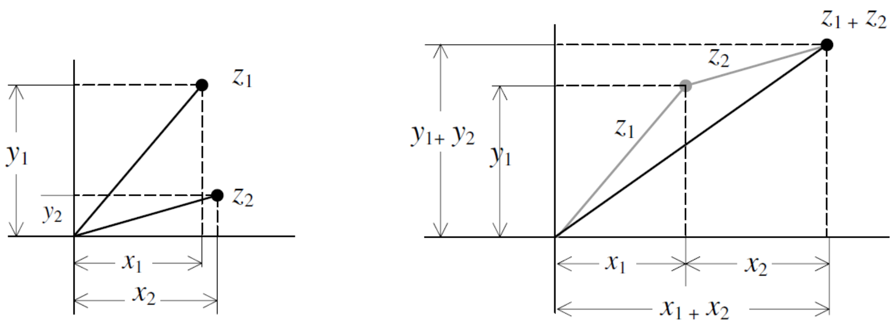


Figure 1.2: Vectors

We see that the sum of complex numbers results in the same line in the complex plane as the sum of two vectors in the x–y plane.

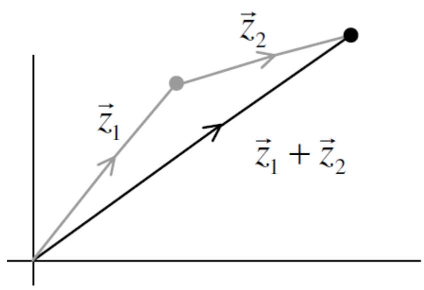


Figure 1.3: The sum of two vectors

There are two possible ways to subtract two vectors, as shown in the figure. The direction associated with each difference vector makes vector subtraction unambiguous.

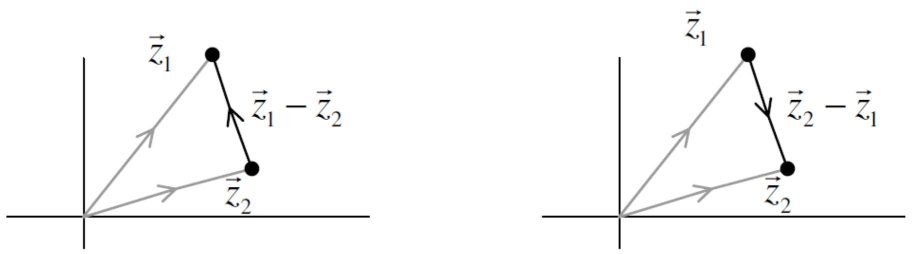


Figure 1.4: Difference of two vectors

The Triangle Inequality

Using the geometric meaning of addition of complex numbers, and the well known result from Euclidean geometry that the sum of the lengths of any two sides of a triangle is at least as big as the length of the third side, we obtain the following *triangle inequality* for any $z_1, z_2 \in \mathbb{C}$:

$$\boxed{|z+w| \leq |z| + |w|}$$

Proof: For any $z, w \in \mathbb{C}$ we have :

$$\begin{aligned}
|z+w|^2 &= (z_+w)\overline{(z+w)} = (z_+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + (z\overline{w} + w\overline{z}) + w\overline{w} \\
&= |z|^2 + (z\overline{w} + \overline{z\overline{w}}) + |w|^2 \\
&= |z|^2 + 2\Re(z\overline{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\overline{w}| + |w|^2 \\
&= |z|^2 + 2|z||\overline{w}| + |w|^2 \\
&= |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}$$

Now take positive square root.

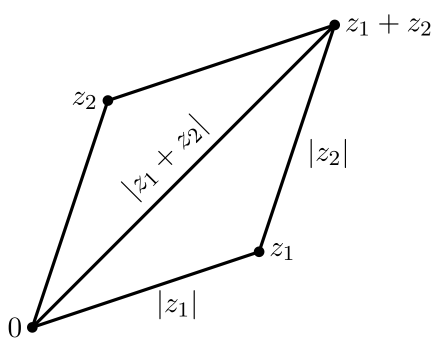


Figure 1.5: Triangle inequality

Remark.

- The triangle inequality can also be verified analytically by using the Cauchy-Schwarz inequality.
- It we replace w by $-w$ we get $|z \pm w| \leq |z| + |w|$
- By applying the triangle inequality to $z = (z - w) + w$, we obtain $|z| \leq |z - w| + |w|$. Subtracting $|w|$, we obtain this very useful inequality,

$$\boxed{|z \pm w| \geq ||z| - |w||.}$$

The Argument of a Complex Number

The **argument** of a complex number denoted $\arg(z)$, is the angle that the vector with tail at the origin and head at $z = x + iy$ makes with the positive x-axis see figure (1.6).

Note that the argument is defined for all **nonzero** complex numbers and is only determined up to an additive integer multiple of 2π. That is, the argument of a complex number is the **infinite** set of values:

$$\arg z = \arg(x + iy) = \tan^{-1}(y/x) = \theta + 2k\pi, \quad \text{where} \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The complex valued-function $w = f(z) = \arg(z)$ is an infinite valued function, because for each $z \in \mathbb{C}$ we may have an infinite number of distinct values of $\arg z$. Such functions are known as **multivalued**.

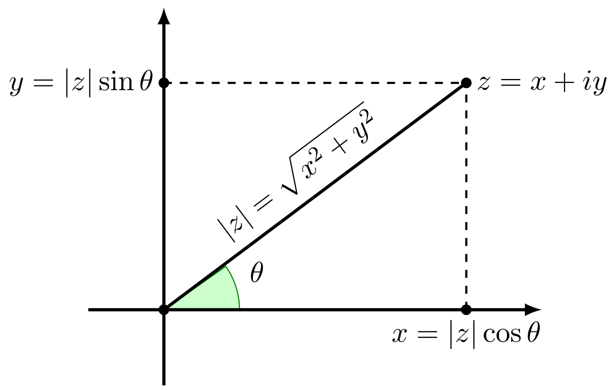


Figure 1.6: Polar form of a complex number

The **principal argument** of a non zero complex number denoted by $\operatorname{Arg} z$ is the unique angle in the set $\arg(z)$ which lies in $(-\pi, \pi]$. We can define $\arg z$ in terms of $\operatorname{Arg} z$ as follows:

$$\theta = \arg z \equiv \operatorname{Arg} z + 2k\pi \quad \text{where} \quad k = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Note that $\operatorname{Arg} z$ is a **single-valued** function of z.

We can now write the **polar form** of a complex number:

$$\boxed{z = x + iy = |z|(\cos\theta + i\sin\theta)}$$

Remarks.

- A single-valued function $w = f(z)$ yields one value w for a given complex number z.
- A multivalued function admits more than one value w for a given z.
- $\arg(z)$ of any (non-zero) complex number has infinitely many possible values.
- The function $\arg(z)$ is the simplest example of a multi-valued function.
- The zero complex number $0 = |z|e^{i\theta}$ has $|z| = 0$ and $\theta = \arg z$ is arbitrary.
- The function $\text{Arg}(z)$ is a single-valued function called a branch of $\arg(z)$.
- We can define other single valued branches of $\arg(z)$ as $\text{Arg}_t(z);\ t \leq \text{Arg}_t < t + 2\pi$.

Example. Find $\text{Arg}(z), \text{Arg}_0(z), \text{Arg}_{5\pi}(z)$ and $\arg(z)$ if $z = 1 - i$.

Solution.

- $\text{Arg}(1-i) = -\dfrac{\pi}{4}$
- $\text{Arg}_0(1-i) = \dfrac{7\pi}{4}$
- $\text{Arg}_{5\pi}(1-i) = 5\pi + \dfrac{3\pi}{4} = \dfrac{23\pi}{4}$
- $\arg(1-i) = -\dfrac{\pi}{4} + 2k\pi; k \in \mathbb{Z}$

We list below a few properties of arguments which the reader should prove.
Properties. If $z, z_1, z_2 \neq 0$ we have

- $\arg(z_1 z_2) = \arg(z_2) + \arg(z_2)$
- $\arg(z_1/z_2) = \arg(z_2) - \arg(z_2)$
- $\arg(1/z) = \arg(\overline{z}) = -\arg(z)$
- $\arg(\overline{z}) = \arg(\overline{z}) = -\arg(z)$
- $\text{Arg}(z_1 z_2) = \text{Arg}(z_2) + \text{Arg}(z_2) + 2k\pi,\ k = 0, \pm 1$

where each formula is understood as a set equality and hold modulo adding integral multiples of 2π.

Warning ! The reader should verify the following:

- $\arg(z^2) = \arg(z) + \arg(z) \neq 2\arg(z)$
- $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_2) + \text{Arg}(z_2)$
- Unlike real numbers, the complex numbers are not ordered. So inequalities, such as $z_1 \geq z_2$ or $z_1 < z_2$, do not make sense in $\mathbb{C}$ unless $z_1, z_2 \in \mathbb{R}$.

1.2 Euler's Formula and Polar form of a Complex Number

Named after Leonhard Euler, is a mathematical formula in complex analysis that establishes the fundamental relationship between the trigonometric functions and the complex exponential function.
As seen in the real analysis courses the Euler number is given by

$$\boxed{e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}}$$

and Euler's formula states that, for any real number $\theta \in \mathbb{R}$,

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}(\theta)}$$

where e is the base of the natural logarithm, i is the imaginary unit, and cos and sin are the trigonometric functions cosine and sine respectively, with the argument θ given in radians.

Polar Form of a Complex Number

Let z be any non-zero complex number then the **polar form** z is given by:

$$\boxed{z = x + iy = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} = |z|\text{cis}\,(\theta)}$$

where θ is the argument of z. Consequently we have

$$\overline{z} = x - iy = |z|(\cos\theta - i\sin\theta) = |z|e^{-i\theta} = |z|\text{cis}(-\theta)$$

Note that

$$\boxed{z = x + iy = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta} = |z|e^{i\arg(z)} = |z|e^{i\,\text{Arg}(z)}e^{i2k\pi} = |z|e^{i\,\text{Arg}(z)}}$$

Polar form of fundamental complex numbers:

(1) $1 = e^{i0} = e^{i2k\pi};\ k \in \mathbb{Z}$

(2) $i = e^{i\pi/2}$

(3) $-1 = e^{i\pi}$

(4) $-i = e^{-i\pi/2}$

(5) $1 \pm i = \sqrt{2}e^{\pm i\pi/4}$

(6) $-1 \pm i = \sqrt{2}e^{\pm i3\pi/4}$

(7) $(1 \pm i\sqrt{3}) = 2e^{\pm i\pi/3}$

(8) $(-1 \pm i\sqrt{3}) = 2e^{\pm 2i\pi/3}$

(9) $(\sqrt{3} \pm i) = 2e^{\pm i\pi/6}$

(10) $(-\sqrt{3} \pm i) = 2e^{\pm 5i\pi/6}$

Properties. For all $\theta, \theta_1, \theta_2 \in \mathbb{R}; k \in \mathbb{Z}$ we have :

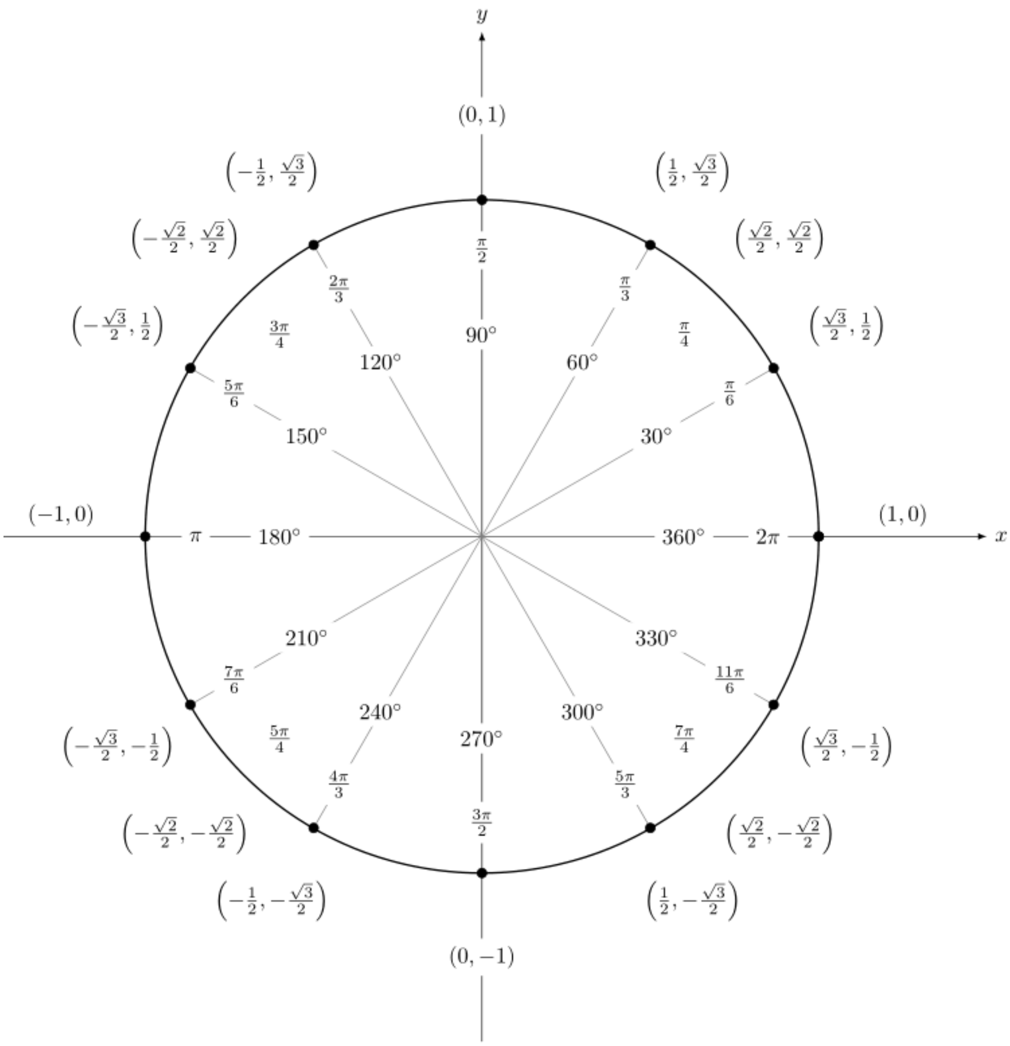


Figure 1.7: Fundamental Trigonometric Angles

(1) $|e^{i\theta}| = 1$

(2) $(e^{i\theta})^k = e^{ik\theta}$

(3) $e^{i2k\pi} = 1$

(4) $e^{i(\theta+2k\pi)} = e^{i\theta}$

(5) $1/e^{i\theta} = e^{-i\theta} = \overline{e^{i\theta}}$

(6) $e^{i\theta_1} \cdot e^{i\theta_1} = e^{i(\theta_1+\theta_2)}$

(7) $e^{i\theta_1}/e^{i\theta_1} = e^{i(\theta_1-\theta_2)}$

(8) $e^{i(2k+1)\pi} = -1$

Proposition. Let $z_k = |z_k|e^{i\theta_k}$ for $k \in \mathbb{N}, n \in \mathbb{Z}$ and $z = |z|e^{i\theta}$. Then we have the following:

$$z_1 z_2 = |z_1||z_2|\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} = |z_1||z_2|e^{i(\theta_1+\theta_2)} \tag{1}$$

$$\frac{z_1}{z_2} = \left|\frac{z_1}{z_2}\right|\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\} = \left|\frac{z_1}{z_2}\right|e^{i(\theta_1-\theta_2)} \tag{2}$$

$$\begin{aligned} z_1 \cdots z_m &= |z_1|\cdots|z_m|\{\cos(\theta_1+\cdots+\theta_m) + i\sin(\theta_1+\cdots+\theta_m)\} \\ &= |z_1|\cdots|z_m|e^{i(\theta_1+\cdots+\theta_m)} \end{aligned} \tag{3}$$

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = (\cos n\theta + i\sin n\theta) \tag{4}$$

$$z^n = |z|^n(\cos\theta + i\sin\theta)^n = |z|^n(\cos n\theta + i\sin n\theta) = |z|^n e^{in\theta} \tag{5}$$

Equation (5) is the famous **De Moivre's formula**.

Remark: Using Euler formula we have the following:

(1) $e^{i\theta} = \cos\theta + i\sin\theta$

(2) $e^{-i\theta} = \cos\theta - i\sin\theta$

(3) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

(4) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(5) $e^{in\theta} = \cos n\theta + i\sin n\theta$

(6) $e^{-in\theta} = \cos n\theta - in\sin\theta$

(7) $\cos n\theta = \dfrac{e^{in\theta} + e^{-in\theta}}{2}$

(8) $\sin\theta = \dfrac{e^{in\theta} - e^{-in\theta}}{2i}$

Example. Find $w = (1 - i\sqrt{3})^{12}$ and $z = (1 - i\sqrt{3})^{-1}$ in the form $a + ib$.
Solution. It is easy to see that $(1 - i\sqrt{3}) = 2e^{-i\pi/3}$, thus

$$w = 2^{12}e^{-i12\pi/3} = 2^{12}e^{-4i\pi} = 2^{12} = 2^{12}(\cos 4\pi - i\sin 4\pi) = 2^{12}.$$

$$z = 2^{-1}e^{+i\pi/3} = 2^{-1}(\cos(\pi/3) + i\sin(\pi/3)) = (1 + i\sqrt{3})/4.$$

The Binomial Theorem. For $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$(z_1+z_2)^n = \sum_{k=0}^{n} z_1^{n-k}z_2^k = z_1^n + \binom{n}{1}z_1^{n-1}z_2 + \binom{n}{2}z_1^{n-2}z_2^2 + \cdots + \binom{n}{k}z_1^{n-k}z_k + \cdots + z_2^n.$$

For all $n \in \mathbb{N}$ we have :

$$\boxed{(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{i(n\theta)} = \cos(n\theta) + i\sin(n\theta).}$$

This formula can help us find $\cos(n\theta)$ and $\sin(n\theta)$ in terms of powers of $\sin\theta$ and $\cos\theta$.

Example. Find $\cos 3\theta$ and $\sin 3\theta$ in terms of powers $\cos\theta$ and $\sin\theta$.

$$\begin{aligned}\cos(3\theta) + i\sin(3\theta) &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

Hence we have

$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Example. Use Euler formula to linearize $\cos^3 x$.
Solution.

$$\begin{aligned}\cos^3 x &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 \\ &= 2^{-3}(e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}) \\ &= 2^{-3}(e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}) \\ &= 2^{-3}(2\cos 3x + 6\cos x) \\ &= 2^{-2}(\cos 3x + 3\cos x)\end{aligned}$$

If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ we have

$$\boxed{z^n = [r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)].}$$

1.3 $n-$th roots of a complex number

Let $w = r(\cos\theta + i\sin\theta)$ be a nonzero complex number and n be a positive integer. Then there are n nth roots of w, defined to be the set of complex numbers

$$w^{1/n} = \{z \in \mathbb{C} : z^n = w\}$$

and given for $k = 0, 1, \ldots, n-1$ by

$$\boxed{z_k = r^{1/n}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right].}$$

Equivalently we can write

$$\boxed{w^{1/n} = \{r^{1/n}e^{i(\theta+2\pi k)/n} : k = 0, 1, \ldots n-1\}.}$$

If $w = 1$, we get the nth **roots of unity** which is the set of complex numbers z such that

$$1^{1/n} = \{z \in \mathbb{C} : z^n = 1\}$$

and given for $k = 0, 1, \ldots, n-1$, by

$$z_k = e^{2ik\pi/n} = \left[\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right].$$

These values are equally spaced points on the unit circle in the complex plane.

So we can write

$$\boxed{1^{1/n} = \{e^{i2\pi k/n} : k = 0, 1, \ldots n-1\}.}$$

Example. The 6 th-roots of unity are

$$1^{1/6} = \{e^{2ik\pi/6} : k = 0, 1, ..., 6\} = \left\{\pm 1, \frac{1 \pm i\sqrt{3}}{2}, \frac{-1 \pm i\sqrt{3}}{2}\right\}$$

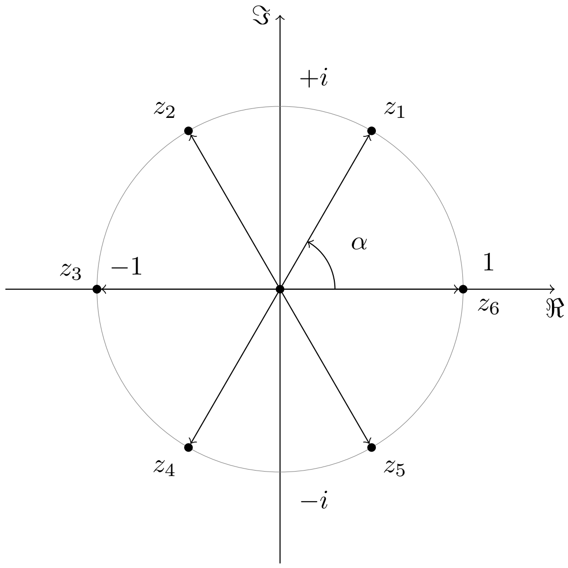


Figure 1.8: The 6 th-roots of unity

Remark.
We have seen earlier that $\arg z$ is an infinite valued function of z and above we saw that for $n \in \mathbb{N}$ the function $z^{1/n}$ is n–valued function of the complex variable z. This type of multivalued functions is specific to complex analysis and is not known in the real case.

1.4 Point at Infinity and the Stereographic Projection

Complex infinity. In real variables, there are only two ways to get to infinity. We can either go up to approach $+\infty$ or down to approach $-\infty$ on the number line. Thus signed infinity makes sense. In the complex plane there are an infinite number of ways to approach infinity. We stand at the origin, point ourselves in any direction and go straight. We could walk along the positive real axis and approach infinity via positive real numbers.

We could walk along the positive imaginary axis and approach infinity via pure imaginary numbers. We could generalize the real variable notion of signed infinity to a complex variable notion of directional infinity, but this will not be useful for our purposes. Instead, we introduce complex infinity or the point at infinity as the limit of going infinitely far along any direction in the complex plane. The complex plane together with the point at infinity form the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Stereographic projection determines a one-to-one correspondence between the unit sphere in $\mathbb{R}^3$ minus the north-pole, S, and the complex plane via the correspondence

$$z \leftrightarrow \frac{x_1 + ix_2}{1 - x_3},$$
$$x_1 = \frac{2\Re z}{1 + |z|^2}, \quad x_2 = \frac{2\operatorname{Im} z}{1 + |z|^2}, \quad x_3 = \frac{|z|^2 - 1}{|z| + 1}.$$

The origin is mapped to the south pole. The point at infinity, $|z| = \infty$, is mapped to the north pole. In the stereographic projection, circles and lines in the complex plane are mapped to circles on the unit sphere. If we define $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, then we have a one-to-one correspondence between S and $\mathbb{C}_\infty$. This allows us to define a metric on $\mathbb{C}_\infty$, which is given by

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}, \qquad d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

1.5 Topology of complex plane

The concepts in ordinary calculus in the setting of $\mathbb{R}$, like convergence of sequences, or continuity and differentiability of functions, all rely on the notion of closeness of points in $\mathbb{R}$. For example, when we talk about the convergence of a real sequence $(c_n), n \in \mathbb{N}$ to its limit $L \in \mathbb{R}$, we mean that given any positive ϵ , there is a large enough index N such that beyond that index, the corresponding terms c_n all have a distance to L which is at most ϵ. This "distance of c_n to L" is taken as $|c_n - L|$, and this is the length of the

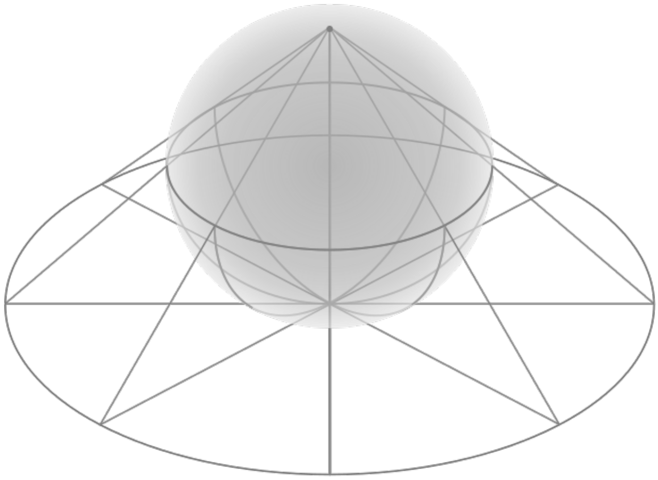


Figure 1.9: Stereographic projection.

line segment joining the numbers c_n and L on the real number line. Now in order to do calculus with complex numbers, we need a notion of distance $d(z_1, z_2)$ between pairs of complex numbers $d(z_1, z_2)$, and the first order of business is to explain what this notion is.

Metric on $\mathbb{C}$. Since $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$, we use on $\mathbb{C}$ the Euclidean distance . Hence, for $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we define the distance by:

$$d(z_1, z_2) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2|. \tag{6}$$

$(\mathbb{C}, |\cdot|)$ is a complete metric space (Banach space).

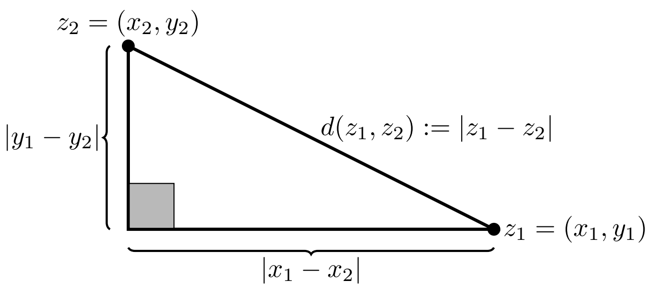


Figure 1.10: Pythagoras theorem

The equation of the circle centered at $a = (a_1, a_2)$ with radius r is given by :

$$(x - a_1)^2 + (y - a_2)^2 = r^2;$$

if we let $z = (x, y)$ and use equation (6), then the equation of the circle can simply be written as

$$|z - a|^2 = r^2 \Leftrightarrow |z - a| = r$$

hence in $\mathbb{C}$ we can define the equation of the the points on the circle centered at a and with radius r by

$$C(a;r) = \{z \in \mathbb{C} : |z-a| = r\}.$$

Open discs, open sets, closed sets and compact sets

(1) **Open disc.** The set $D(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$ is called the open disc (open ball) centered at $a \in \mathbb{C}$ and with radius $r > 0$.

(2) **Unit disc.** $\mathbb{D} = D(0;1) = \{z \in \mathbb{C} : |z| < 1\}$ is called the open unit disc of $\mathbb{C}$.

(3) **Open set.** A set $U \subset \mathbb{C}$ is open in $\mathbb{C}$ if for all $z \in U$ there exists $r > 0$ such that $D(z;r) \subset U$. The half planes $\{z \in \mathbb{C} : \Re z > a\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z > b\}$ are open in $\mathbb{C}$.

(4) **Closed set.** A set $F \subset \mathbb{C}$ is **closed** in $\mathbb{C}$ if its complement $\mathbb{C} \setminus F$ is open in $\mathbb{C}$.
If $F \subset \mathbb{C}$ and $\partial F \subset F = \varnothing$ then F is closed in $\mathbb{C}$.
A set is closed if it contains all its boundary points.
$\overline{D}(a;r) = \{z \in \mathbb{C} : |z-a| \leq r\}$ is closed in $\mathbb{C}$, and called a closed disc.
The annulus $A(a,r,R) = \{z \in \mathbb{C} : r \leq |z-a| \leq R\}$ is closed in $\mathbb{C}$.
A set $F \subset \mathbb{C}$ is closed if and only if every convergent sequence (z_n) in F has a its limit in F, $z_n \to z \in F$.

(5) **Adherence or Closure of a set.** If $S \subset \mathbb{C}$ then the set $\overline{S} = S \cup \partial S$ is the adherence or the closure of S. The adherence of the open disc $D(a;r)$ is the closed disc $\overline{D}(a;r)$.

(6) **Bounded set.** A set $S \subset \mathbb{C}$ is **bounded** if there exists $M > 0$ such that $|z| < M$ for all $z \in S$. Equivalently we say that S is bounded if $S \subset D(0;r)$ for some $r > 0$.
The set $|z| < 4$ is bounded, but $\{z \in \mathbb{C} : \Re z > 0\}$ is not.

(7) **Compact set .** A set $K \subset \mathbb{C}$ is **compact** in $\mathbb{C}$ if it is bounded and closed in $\mathbb{C}$. The set $|z| \leq 4$ is compact, but $|z| < 4$ is not.

(8) **Connected by arcs.** An open set $S \subset \mathbb{C}$ is said to be connected by arcs if any two points can be connected by a path that is entirely in S.

(9) **Connected set.** An open set $S \subset \mathbb{C}$ is said to be **connected** if it can not be the union of two non-empty disjoint open sets.
Any set of $\mathbb{C}$ connected by arcs is connected.
The open unit disc $|z| < 1$ and the annulus $1 < |z| < 2$ are connected because they are connected by arcs.

(10) **Region.** A region is an open polygonally-connected set S together with all, some or none of its boundary points. We assume polygonal-connectedness to avoid infinite length paths and fractal-like open sets.

(11) **Simply connected sets.** A connected by arcs set $S \subset \mathbb{C}$ is said simply connected if any closed path on S can be continuously reduced (by homotopy) to a point. Intuitively, one can shrink the closed path until it forms one point. It is a region which contains no holes.
The disc $|z| < 1$ is simply connected, but the crown $1 < |z| < 2$ is not. The private plan of a $\mathbb{C} \setminus \{z_0\}$ point is connected but not merely connected. In other words, a simply connected set does not have "**holes**". If it has holes it is called **multi-connected**. The annulus is an example of a multi-connected region.

(12) **Domain.** A non-empty open and connected set D in $\mathbb{C}$ is called a **domain** or a **open region**.

Argand Diagrams

Example: Indicate graphically, on a single Argand diagram, the sets of values of z determined by the following relations:

(a) Point $z = 1 - 2i$
(b) Line $|x + 1 + i| = |z - 1 - i|$
(c) Circle $|z - 1 - i| = 1$
(d) Disc $|z - 1 - i| < 1$
(e) Ellipse $|z + i| + |z + 2i| = 2$
(f) Annulus $1 \leq |z + 3| \leq 2$
(g) Strip $3 \leq \Re z \leq 5$
(h) Ray $\operatorname{Arg} z = -3\pi/4$

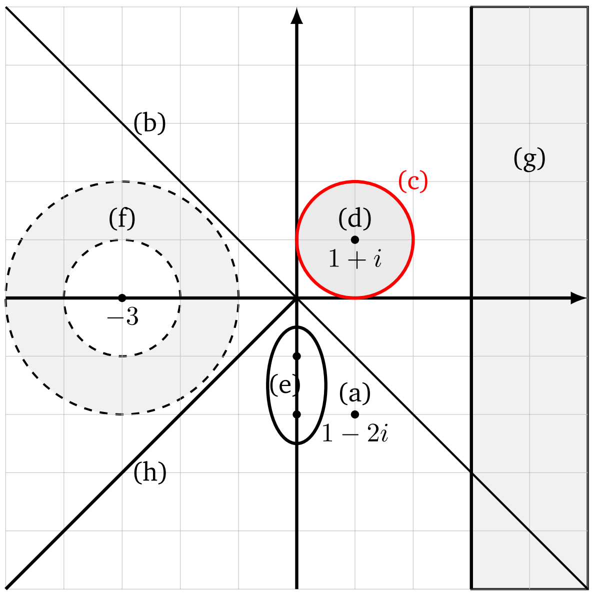


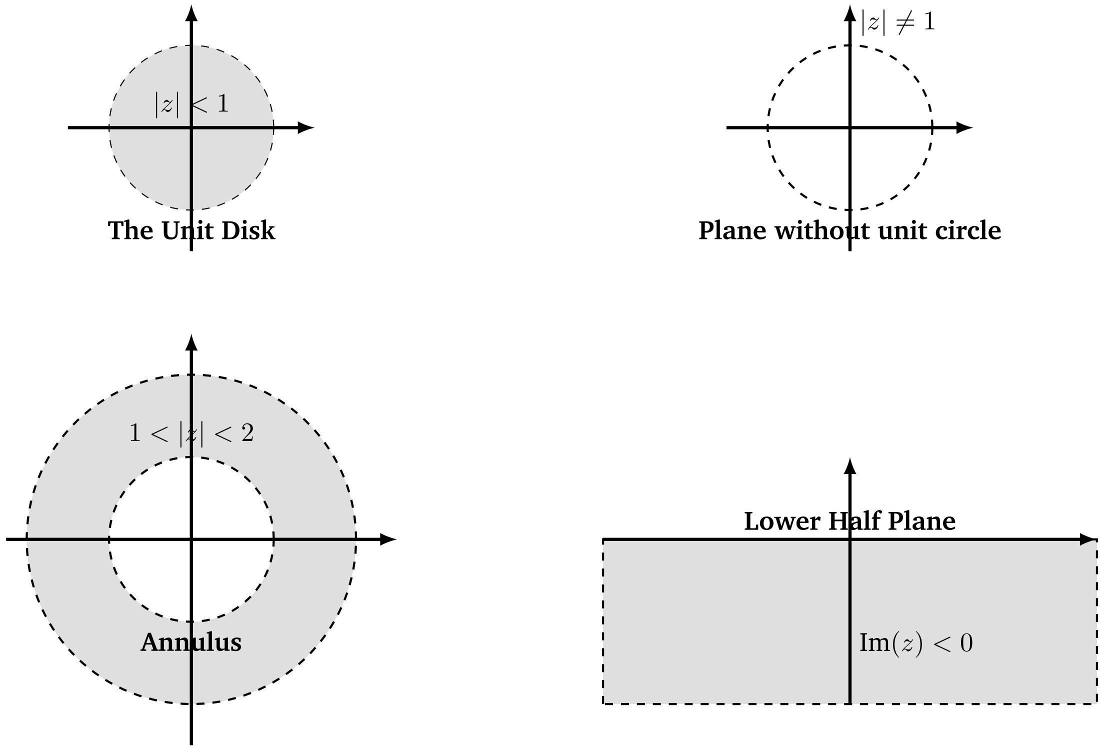
Figure 1.11: Argand Diagrams

Examples of Planar Sets

- **Open and Closed Discs:**
  Disc (d) $|z - 1 - i| < 1$ is an open disc.

Disc (i) $|z-1-i| \leq 1$ is a closed disc.
Disc (d) is the interior of disc (i).
The exterior of disc (d) $|z-1-i| \geq 1$ is closed.

- **Regions:**
  The disc (d), the annulus (f) and the strip (g) are regions.
  The open elliptical disc (j) $|z-i|+|z-2i|<2$ is also a region.
- **Boundaries:**
  The boundary of Disc (d) is the Circle (c).
  The boundary of Annulus (f) is the union of the circles $|z+3|=1$ and $|z+3|=2$.
  The boundary of the Strip (g) is the union of the lines $\Re z=3$ and $\Re z=5$.
- **Open and Closed Sets:**
  The planar sets (d) and the Elliptical Disc (j) $|z-i|+|z-2i|<2$ are open.
  The sets (a), (b), (c), (e), (f), (g) are closed.
  The Ray (h) and the strip $3<\Re z \leq 5$ are neither open nor closed.
  Note the Ray (h) does not contain the boundary point at the origin since $\operatorname{Arg} z$ is not defined there.
- **Bounded and Compact Sets:**
  The sets (a), (c), (d), (e), (f) are bounded.
  The sets (b), (g), (h) are unbounded.
  The sets (a), (c), (e), (f) are compact.
- **Connected Open Sets:** The Disc (d), the open Elliptical Disc (j) $|z-i|+|z-2i|<2$ and the interiors of the Annulus (f) and Strip (g) are connected.
  The disjoint union of the open sets (d) and (j) is not connected.
  Likewise the set $\mathbb{C} \setminus \{|z|=1\}$ is not connected.
  The Annulus (f) is connected but not simply connected because loops around the hole cannot be continuously shrunk to zero.
- **Domains:**
  The unit disc $D(0,1)=\{z \in \mathbb{C} : |z|<1\}$, the annulus $1<|z|<2$ and the half-plane $\{z \in \mathbb{C} : \operatorname{Im} z<0\}$ are domains, but $S=\{z \in \mathbb{C} : |z| \neq 1\}$ is not a domain because it is not connected.


$|z| < 1$
The Unit Disk
$|z| \neq 1$
Plane without unit circle
$1 < |z| < 2$
Annulus
Lower Half Plane
$\mathrm{Im}(z) < 0$

2 Complex Differentiation and Analytic Functions

Augustin Louis Cauchy (1789–1857)

Georg Friedrich Bernhard Riemann (1826–1866)

Nothing in our experience suggests the introduction of [complex numbers]. Indeed, if a mathematician is asked to justify his interest in complex numbers, he will point, with some indignation, to the many beautiful theorems in the theory of equations, of power series, and of analytic functions in general, which owe their origin to the introduction of complex numbers. The mathematician is not willing to give up his interest in these most beautiful accomplishments of his genius.

– Eugene Paul Wigner

In real analysis, one studies (rigorously) calculus in the setting of real numbers. Thus one studies concepts such as the convergence of real sequences, continuity of real-valued functions, differentiation and integration. Based on this, one might guess that in complex analysis, one studies similar concepts in the setting of complex numbers. This is partly true, but it turns out that up to the point of studying differentiation, there are no new features in complex analysis as compared to the real analysis counterparts. But the subject of complex analysis departs radically from real analysis when one studies differentiation. Thus, complex analysis is not merely about doing analysis in the setting of complex numbers, but rather, much more specialized:

Complex analysis is the study of **"complex differentiable"** functions.

2.1 Complex valued functions

Real-valued functions of a real variable can be visualized by graphing them in the plane $\mathbb{R}^2$. The graph of a complex-valued function $f(z)$ of a complex variable z requires four (real) dimensions. To visualize the behavior of $w = f(z)$, we create two planes, a z-plane for the domain space and a w-plane for the range space. We then view $f(z)$ as a mapping from the z-plane to the w-plane, and we analyze how various geometric configurations in the z-plane are mapped by $w = f(z)$ to the w-plane. Which geometric configurations in the z-plane to consider depends very much on the specific function $f(z)$.

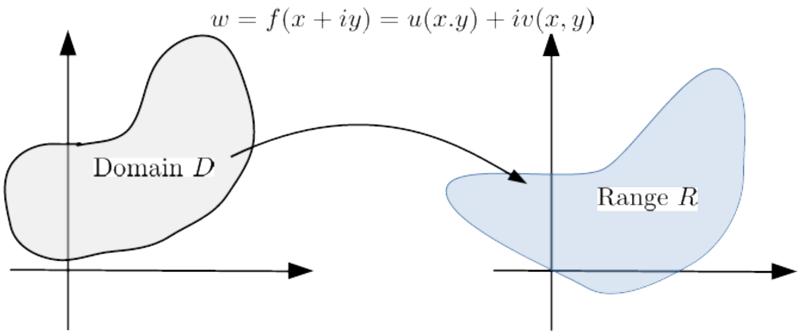


Figure 2.1: The mapping $w = f(z)$

A complex-valued function f of the complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We write $w = f(z)$ and call w the image of z under f. The set D is called the domain of f, and the set of all images $\{w = f(z) : z \in D\}$ is called the range of f. We can define the domain to be any set that makes sense for a given rule. It could be the domain of definition of f or any subset of it. Determining the range for a function defined by a formula is not always easy, but we will see plenty of examples later on. In some contexts functions are referred to as mappings or transformations. When the context is obvious, we omit the phrase *complex-valued*, and simply refer to a function f, or to a complex function f.

Just as z can be expressed by its real and imaginary parts, $z = x + iy$, we can write $f(z) = w = u + iv$, where u and v are the real and imaginary parts of w, respectively. Doing so gives us the representation

$$w = f(z) = f(x,) = f(x + iy) = u + iv.$$

Because u and v depend on x and y, they can be considered to be real-valued functions of the real variables x and y; that is, $u = u(x, y)$ and $v = v(x, y)$. Combining these ideas, we often write a complex function f in the form

$$f(z) = f(x + iy) = u(x, y) + iv(x, y).$$

Let us look at the simple example of a complex-valued function is given by the formula $w = f(z) = z^2$. We can define the domain to be any set that makes sense for a given rule, so for $w = f(z) = z^2$, we could have the entire complex plane $\mathbb{C}$ for the domain D, or we might artificially restrict the domain to some set such as the unit disc $\mathbb{D}$. Using the binomial formula, we obtain

$$w = f(z) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$$

so that $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

2.2 Single and Multiple-Valued Functions

Consider the complex valued function $w = f(z)$. If only one value of w corresponds to each value of z , we say that w is a single-valued function of z or that $f(z)$ is single-valued. If more than one value of w corresponds to each value of z , we say that w is a multiple-valued or many-valued function of z. A multiple-valued function can be considered as a collection of single-valued functions, each member of which is called a **branch** of the function. It is customary to consider one particular member as a principal branch of the multiple-valued function and the value of the function corresponding to this branch as the principal value.

Example.

(a) If $w = z^2$, then to each value of z there is only one value of w .
Hence, $w = z^2$ is a single-valued function of z .

(b) If $w^2 = z$, then to each value of z there are two values of w .
Hence, $w^2 = z$ defines a multiple-valued (in this case two-valued) function of z .

Whenever we speak of function, we shall, unless otherwise stated, assume single-valued function.

2.3 Convergence and Continuity

We can also talk about convergent sequences in $\mathbb{C}$. A sequence $(z_n)_{n\in\mathbb{N}}$ is said to be convergent with limit L if for every $\epsilon > 0$, there exists an index $N \in \mathbb{N}$ such that for every $n > N$, there holds that $|z_n - L| < \epsilon$. It follows from the triangle inequality that for a convergent sequence the limit is unique, and we write $\lim\limits_{n\to\infty} z_n = L$.

Example. Let z be a complex number with $z < 1$.

Then the sequence $(z^n)_{n\in\mathbb{N}}$ converges to 0. Indeed, $|z^n - 0| = |z^n| = |z|^n \to 0$. Let S be a subset of $\mathbb{C}$, $z_0 \in S$ and $f : S \to \mathbb{C}$. Then f is said to be continuous at z_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $z \in S$ satisfies $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \epsilon$. f is said to be continuous in S if it is continuous for every $z \in S$.

One can also give a characterization of continuity at a point in terms of convergent sequences.

$f : S \to \mathbb{C}$ is continuous at $z_0 \in S$ if and only if for every sequence $(z_n)_{n\in\mathbb{N}}$ in S convergent to z_0, then the sequence ($(f(z_n))_{n\in\mathbb{N}}$ is convergent to $f(z_0)$.

Example. Complex conjugation is continuous, $f : \mathbb{C} \to \mathbb{C}$ such that $f(z) = \bar{z}$ is continuous. Indeed, we have $|\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0|$ for all $z, z_0 \in \mathbb{C}$.

This shows that complex conjugation is continuous at each $z_0 \in \mathbb{C}$, and so it is a continuous mapping. This is geometrically obvious, since complex conjugation is just reflection in the real axis, and so the image stays close to the reflected point if we are close to the point. Since $\overline{(\bar{z})} = z$ for all $z \in \mathbb{C}$, complex conjugation is its own inverse. So complex conjugation is invertible with a continuous inverse. Thus complex conjugation gives a homeomorphism (that is, a continuous bijective mapping with a continuous inverse) from $\mathbb{C}$ to $\mathbb{C}$.

2.4 Complex differentiability and analyticity

Definition.

(1) Let D be an open subset of $\mathbb{C}$, $f : D \to \mathbb{C}$ and $a \in D$. Then f is said to be complex differentiable at a if there exists a complex number L such that

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a} = L = \lim_{\Delta z\to 0}\frac{f(a+\Delta z)-f(a)}{\Delta z}$$

We denote this L (which can be shown to be unique) by $f'(a)$ or $\dfrac{df}{dz}(a)$.

(2) A function $f : D \to \mathbb{C}$ is said to be analytic (holomorphic) at $z = a$ if it is differentiable in a neighborhood of a .

(3) A function analytic at every point of complex plane $\mathbb{C}$ is called entire.

(4) We say that f has a singularity at $z = a$ if f is not analytic at $z = a$.

Remark.
The key feature of the definition of differentiability is that the limiting value $f'(z)$ of the difference quotient must be independent of how z converges to a. On the real line, there are only two directions to approach a limiting point thats is either from the left or from the right. These lead to the concepts of left and right handed derivatives and their equality is required for the existence of the usual derivative of a real function. In the complex plane, there are an infinite variety of directions for the variable z to approach the point a, and the definition requires that all of these "directional derivatives" must agree. This is the reason for the more severe restrictions on complex derivatives, and, in consequence, the source of their remarkable properties.

Remark.
Note that if f is differentiable at every point of an open set in $\mathbb{C}$ it is automatically analytic; in fact, it is automatically infinitely differentiable. This is of course vastly different from the real case.

Example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$. We show that f is entire. Note that for every $a \in \mathbb{C}$ we have:

$$\lim_{z\to a}\frac{f(z)-f(a)}{z-a} = \lim_{z\to a}\frac{z^2-a^2}{z-a} = \lim_{z\to a}(z+a) = 2a = f'(a).$$

Hence f is entire and $f'(z) = 2z$.

Example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \bar{z}$. We show that f is differentiable nowhere. Note that for every $a \in \mathbb{C}$ we have:

$$\lim_{\Delta z \to 0} \frac{f(a+\Delta z) - f(a)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{a+\Delta z} - \bar{a}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \begin{cases} +1 & \text{if } \Delta y = 0 \\ -1 & \text{if } \Delta x = 0 \end{cases}$$

Hence f is nowhere differentiable and hence nowhere analytic.

Theorem. If $f(z)$ is differentiable at z, then $f(z)$ is continuous at z.
This follows from the sum and product rules for limits. We write

$$f(z) = f(z) + \frac{f(z) - f(a)}{z - a}(z - a)$$

Since the difference quotient tends to $f'(a)$ and $(z - a)$ tends to 0 as $z \to a$ then consequently, $f(z) \to f(a)$ as $z \to a$.

Example. The function $f(z) = |z|^2$ is only differentiable at 0 and is analytic nowhere.

We have the following implications.

Analyticity $\Longrightarrow$ $\mathbb{C}$–Differentiability $\Longrightarrow$ Continuity

Definition of differentiability at a point (assumes function is defined in a neighborhood of the point). Most of the consequences of differentiability are quite different in the real and complex case, but the simplest algebraic rules are the same, with the same proofs. First of all, differentiability at a point implies continuity there. If f and g are both differentiable at a point a, then so are $f \pm g$, $f \cdot g$, and, if $g(a) \neq 0$, f/g, and the usual sum, product, and quotient rules hold. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and the chain rule holds. Suppose that f is continuous at a, g is continuous at $f(a)$, and $g(f(z)) = z$ for all z in a neighborhood of a. Then if $g'(f(a))$ exists and is non-zero, then $f'(a)$ exists and equals $1/g'(f(a))$.

Rules for Differentiation. Suppose $f(z), g(z)$, and $h(z)$ are analytic functions of z. Then the following differentiation rules (identical with those of elementary calculus) are valid.

(1) $\dfrac{d}{dz}[f(z) \pm g(z)] = \dfrac{d}{dz}f(z) \pm \dfrac{d}{dz}g(z)$.

(2) $\dfrac{d}{dz}[cf(z)] = c\dfrac{d}{dz}f(z)$ where c is a constant.

(3) $\frac{d}{dz}f(z)g(z) = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z)$

(4) $\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}$

(5) If $w = f(t)$ where $t = g(z)$ then $\frac{dw}{dz} = \frac{dw}{dt}\frac{dt}{dz} = f'[g(z)]g'(z)$.

(6) If $w = f(z)$ has a single-valued inverse f^{-1}, then $z = f^{-1}(w)$, and $\frac{dw}{dz} = \frac{1}{dz/dw}$.

2.5 Cauchy–Riemann Equations

Let $f : D \to \mathbb{C}$ such that $f = u + iv$ and $D \subset \mathbb{C}$ open. We will abuse the notation slightly by writing $f(x, y)$ as an alternative for $f(x + iy)$. Fix a point $z \in D$. We will compute the complex derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

in two different ways, first by letting $z + \Delta z$ tend to z along the horizontal x–axis (that is, $\Delta z = \Delta x$ real), then by letting $z + \Delta z$ tend to z along the vertical imaginary axis (that is, $\Delta z = i\Delta y$ imaginary). This yields two expressions for $f'(z)$, which lead to the Cauchy-Riemann equations.
If $f'(z)$ exists for some $z = x + iy \in D$, then if let $\Delta y = 0$ we get $\Delta z = \Delta x$ and

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{\partial f}{\partial x},$$

and if let $\Delta x = 0$ we get $\Delta z = i\Delta y$ and

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{i\Delta y} = -i\frac{\partial f}{\partial y}.$$

Thus complex-differentiability of f at z implies not only that the partial derivatives of f exist there, but also that they satisfy the **Cauchy–Riemann** equation

$$\boxed{\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.}$$

If $f = u + iv$, then this equation is equivalent to the system also known as the **Cauchy–Riemann** equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},}$$

or simply

$$\boxed{u_x = v_y \quad \text{and} \quad v_x = -u_y.}$$

Theorem. (Cauchy–Riemann) Let $f = u + iv$ be defined on a domain D in the complex plane, where u and v are real-valued. Then $f(z)$ is analytic on D if and only if $u(x, y)$ and $v(x, y)$ have continuous first-order partial derivatives that satisfy the Cauchy-Riemann equations.

Remark.
The theorem can be weakened to say that if f is continuous on D and the partial derivatives exist and satisfy the Cauchy–Riemann equations there (without assuming that the partial derivatives are continuous), then the complex derivative of f exists on D (which is equivalent to f being analytic on D. This is the Looman–Menchoff Theorem.
We do need at least continuity, since otherwise we could take f to be the characteristic function of the coordinate axes.

If f is analytic then its derivative can be written as

$$\boxed{f'(z) = u_x + iu_y = v_y - iu_x}$$

Notice that if $f'(z) = 0$ ten $u_x = u_y = v_x = v_y = 0$ and thus we have the following theorem.

Theorem If $f(z)$ is analytic on a domain D, and if $f'(z) = 0$ on D, then $f(z)$ is constant.

Another convenient notation is to introduce

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

(These are motivated by the equations $x = (z + \bar{z})/2$, $y = (z - \bar{z})/(2i)$, which, if z and $\bar{z}$ were independent variables, would give $\partial x/\partial z = 1/2$, $\partial y/\partial z = -i/2$, etc.) In terms of these, the Cauchy–Riemann equations are exactly equivalent to

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{or} \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = f'(z).}$$

Example. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z^2$. We show that f is entire. Note that $f(x + iy) = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ and hence we have $u_x = 2x = v_y$ and $v_x = 2y = -u_y$ which shows that Cauchy Riemann equations are satisfied and furthermore all partial derivatives are continuous. Hence f is entire and $f'(x + iy) = u_x + iv_x = 2x + i2y = 2z$.

Example. Consider the function $f(z) = \bar{z}$. Since $\dfrac{\partial f}{\partial \bar{z}} = 1 \neq 0$, then f is nowhere analytic.

Example. The function $\log_t z = \ln|z| + i\theta, \quad -t < \theta < t + 2\pi$ is a branch of $\log z$. It is analytic in the indicated domain and its derivative is given by

$$\frac{d}{dz}\log_t z = \frac{1}{z}.$$

In particular the the derivative of the principal logarithm is

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

Example. Let $w = f_1(z) = \sqrt{z}$ be the principal branch of the square root function. Then it is an analytic bijection between the slit plane $\mathbb{C}\setminus(-\infty,0]$ and the open right-half plane $\operatorname{Re} z > 0$. Furthermore its derivative is given by

$$\frac{d}{dz}f_1(z) = \frac{1}{2\sqrt{z}} = \frac{1}{2f_1(z)}.$$

Example: Show that the function $f(z) = e^z = e^x\cos y + ie^x\sin y$ is entire with derivative

$$\boxed{\frac{d}{dz}e^z = e^z}$$

Solution: The first partial derivatives are continuous and satisfy the Cauchy-Riemann equations everywhere in $\mathbb{C}$

$$u_x = v_y = e^x\cos x, \qquad v_x = -u_y = e^x\sin y$$

Hence by the Cauchy-Riemann theorem $f(z) = e^z$ is entire and

$$f'(z) = u_x + iv_x = e^x\cos x + ie^x\sin y = e^z.$$

Example. The functions $\sin(z) = \dfrac{e^{iz} - e^{iz}}{2i}$ and $\cos(z) = \dfrac{e^{iz} + e^{iz}}{2}$ are entire complex valued functions and we have

$$\boxed{\frac{d}{dz}\sin(z) = \cos(z)} \quad \text{and} \quad \boxed{\frac{d}{dz}\cos(z) = -\sin(z)}$$

Example. Discuss where the function $f(x+iy) = (x^2+y) + i(y^2-x)$ is (a) differentiable and (b) analytic.
Solution: Since $u(x,y) = x^2 + y$ and $v(x,y) = y^2 - x$, we have

$$u_x = 2x, \quad v_y = 2y, \quad v_x = -u_y = -1$$

These partial derivatives are continuous everywhere in $\mathbb{C}$. They satisfy the Cauchy-Riemann equations on the line $y = x$ but not in any open region. It follows by the Cauchy-Riemann theorem that $f(z)$ is differentiable at each point on the line $y = x$ but nowhere analytic.

Cauchy-Riemann equations in polar form

Proposition. Let $f(r, \theta) = u(r, \theta) + iv(r, \theta)$ be analytic function at $z_0 = r_0 e^{i\theta_0}$. Then the Cauchy Riemann equations in polar form take the form:

$$ru_r = v_\theta \quad \text{et} \quad rv_r = -u_\theta \tag{1}$$

Proof. Let $z = re^{i\theta}$ où $x = r\cos\theta$, $y = r\sin\theta$, we will have $\theta = \arg z$ and $|z| = r$. It is clear that $f(z) = u(x, y) + iv(x, y)$:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$

that is

$$u_r = u_x \cos\theta + u_y \sin\theta \quad \text{and} \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta. \tag{2}$$

We will also get

$$v_r = v_x \cos\theta + v_y \sin\theta \quad \text{and} \quad v_\theta = -v_x r\sin\theta + v_y r\cos\theta. \tag{3}$$

Since f is analytic at z_0 then the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ are satisfied and (3) becomes

$$v_r = -u_y \cos\theta + u_x \sin\theta \quad \text{et} \quad v_\theta = u_y r\sin\theta + u_x r\cos\theta. \tag{4}$$

Thus we get from (2) and (4) that $ru_r = v_\theta$ and $rv_r = -u_\theta$. ■

Corollary. If $f(z) = f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$, polar for of $f'(z)$ is

$$f'(z) = e^{-i\theta}(u_r + iv_r) = \frac{1}{r}e^{-i\theta}(v_\theta - iv_\theta) \tag{5}$$

Example. Consider the complex valued function $f(z) = \dfrac{1}{z}$ in $\mathbb{C} - \{0\}$. Let $z = re^{i\theta}$ then we will have

$$f(z) = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r} = r^{-1}(\cos\theta - i\sin\theta)$$

and thus we get

$$u(r, \theta) = r^{-1}\cos\theta \quad v(r, \theta) = -r^{-1}\sin\theta$$

the Cauchy-Riemann equations are satisfied since

$$u_r = -r^{-2}\cos\theta = r^{-1}v_\theta \quad \text{et} \quad v_r = r^{-2}\sin\theta = -r^{-1}u_\theta.$$

The derivative est then

$$f'(z) = e^{-i\theta}(u_r - iv_r) = e^{-i\theta}(-r^{-2}\cos\theta + ir^{-2}\sin\theta) = -r^{-2}e^{-2i\theta} = -z^{-2}.$$

2.6 Inverse Mappings and the Jacobian

Let $f = u + iv$ be analytic on a domain D. We may regard D as a domain in the Euclidean plane $\mathbb{R}^2$ and f as a map from D to $\mathbb{R}^2$ with components $(u(x, y), v(x, y))$. The Jacobian matrix of this map is

$$J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

and the determinant of the Jacobian matrix is

$$|J_f| = u_x v_y - u_y v_x = (u_x)^2 + (v_x)^2 = |u_x + iv_x|^2 = |f'(z)|^2.$$

Theorem. Suppose $f(z)$ is analytic on a domain D, $a \in D$ and $f'(a) \neq 0$. Then there is a (small) disc $U \subset D$ containing a such that $f(z)$ is one-to-one on U, the image $V = f(U)$ of U is open, and the inverse function $f^{-1} : V \to U$ is analytic and satisfies

$$(f^{-1})'(f(z)) = 1/f'(z), \quad z \in U.$$

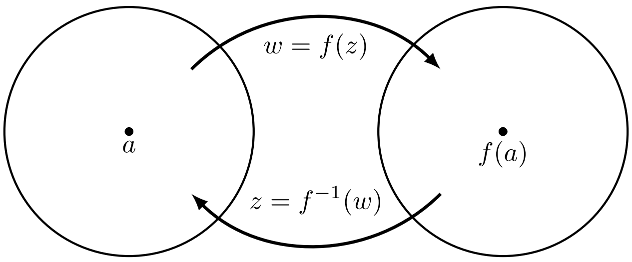


Figure 2.2: Inverse mapping

If we write $w = g(z)$, the above identity becomes

$$\frac{dz}{dw} = \frac{1}{\frac{dw}{dz}}$$

which is the usual formula for remembering the derivative of the inverse function.
Once we know that f^{-1} is analytic, we can easily derive the formula for the derivative

from the chain rule. Since $f^{-1}(f(z))z$, the chain rule yields $(f^{-1})'(f(z))f'(z) = 1$, which gives the above identity of the derivative of the inverse.

Example. The principal logarithm function $w = \operatorname{Log} z$ is a continuous inverse for $z = e^w$ for $-\pi < \arg w < \pi$. Since e^w is analytic and$(e^w)' \neq 0$, the preceding theorem applies, with z and w interchanged. From that theorem we conclude that $\operatorname{Log} z$ is analytic. If we use the chain rule to differentiate

$$z = e^{\log z}$$

we get

$$1 = e^{\operatorname{Log} z}\frac{d}{dz}(\operatorname{Log} z) = z\frac{d}{dz}(\operatorname{Log} z) \Longrightarrow \frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

2.7 Harmonic Functions

A real-valued function $\phi(x, y)$ is said to be harmonic in a domain D if if all its first and second order partial derivatives exist, are continuous and satisfy Laplace's equation

$$\boxed{\phi_{xx} + \phi_{yy} = 0}$$

at each point of D.
In the case of functions of two variables, there is an intimate connection between analytic functions and harmonic functions.
The Laplace equation occurs in many areas of two-dimensional physics including continuum and fluid mechanics, aerodynamics and the heat equation. We see that the solutions to these equations (harmonic functions) are naturally associated with analytic functions.

Theorem (Harmonic Functions).
If $f = u + iv$ is analytic in an open connected domain D , then u and v are harmonic in D.
Proof: Since $f(z)$ is analytic, $u(x, y)$ and $v(x, y)$ are C^∞ (possess continuous partial derivatives of all orders). We will prove this later. In particular, since they are C^2 , the mixed second derivatives are equal

$$(u_x)_y = (u_y)_x, \qquad (v_y)_x = (v_x)_y$$

Substituting for the first partial derivatives from the Cauchy-Riemann equations give

$$v_{yy} = -v_{xx}, \qquad -u_{yy} = u_{xx}.$$

Remark.
(1) The harmonicity of u and v is a simple consequence of the Cauchy-Riemann equations.
(2) The second hypothesis of the theorem is redundant. We will see in page (101) that an analytic function is infinitely differentiable and thus has continuous partial derivatives

of all orders.

Note that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \frac{1}{4}\Delta.$$

This shows that any analytic function is harmonic (equivalently, its real and imaginary parts are harmonic). It also shows that the conjugate of an analytic function, while not analytic, is harmonic.

Let $u(x,y)$ and $v(x,y)$ be two functions harmonic in a domain D that satisfy the Cauchy-Riemann equations at every point of D . Then, $u(x,y)$ and $v(x,y)$ are called harmonic conjugates of each other. Knowing one of them, we can reconstruct the other to within an arbitrary constant.

Example. Show that $u(x,y) = xy$ is harmonic, and find a harmonic conjugate for u.
Solution. Since $u_{xx} = 0 = -u_{yy}$, then u is harmonic. To find a harmonic conjugate v, we solve the Cauchy-Riemann equations.

$$u_x = y = v_y \Longrightarrow v(x,y) = y^2/2 + h(x)$$

where $h(x)$ depends only on x and not on y. Since $u_y = -v_x \Rightarrow h'(x) = -x$ which gives that $h(x) = -x^2/2 + C$ where C est a constant . Thus

$$f(x+iy) = u(x,y) + iv(x,y) = xy + i(y^2/2 - x^2/2 + C) = -iz^2/2 + iC.$$

Example. Does there exist an analytic function on the complex plane whose real part is given by u(x,y) = 3x 2 + xy + y 2 ?
Solution. Clearly, $u_{xx} = 6, u_{yy} = 2$, and hence $u_{xx} + u_{yy} \neq 0$; i.e., u is not harmonic. Thus, no such analytic function exists.

Example Find an analytic function f whose imaginary part is given by $e^{-y}\sin x$.
Solution. Let $v(x,y) = e^{-y}\sin x$. Then it is easy to check that $v_{xx} + v_{yy} = 0$. We have to find a function $u(x,y)$ such that

$$(1) u_x = v_y = -e^{-y}\sin x, \quad (2) u_y = -v_x = -e^{-y}\cos x$$

From (1) we get $u(x,y) = e^{-y}\cos x + \varphi(y)$. Substituting this expression in (1), we obtain

$$-e^{-y}\sin x + \varphi'(y) = -e^{-y}\sin x.$$

Hence, $\varphi'(y) = 0$; i.e., $\varphi(y) = c$ for some constant c .

Thus, $u(x,y) = e^{-y}\cos x + c$ and

$$f(z) = e^{-y}\cos x + c + ie^{-y}\sin x = e^{-y+ix} + c = e^{iz} + c.$$

3 Elementary Complex Valued Functions

The only way to learn mathematics is to do mathematics.
– Paul Halmos (1916-2006; Hungarian-born mathematician)

In this chapter we will see, for some special functions, what happens to regions in the z plane when mapped onto the regions in the w plane.
The graph of a complex-valued function $f(z)$ of a complex variable z requires four (real) dimensions. To visualize the behavior of $w = f(z)$, we create two planes, a z-plane for the domain space and a w-plane for the range space. We then view $f(z)$ as a mapping from the z-plane to the w-plane, and we analyze how various geometric configurations in the z-plane are mapped by $w = f(z)$ to the w-plane. Which geometric configurations in the z-plane to consider depends very much on the specific function $f(z)$.

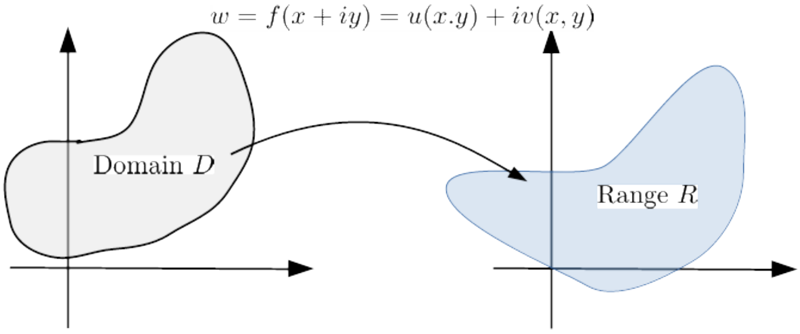


Figure 3.1: The mapping $w = f(z)$

Suppose that D and E are subsets of $\mathbb{C}$. A complex (single-valued) function or mapping $f : D \to E$ of the complex variable z is a rule that assigns to each complex number $z \in D$ one and only one complex number $w = f(z) \in E$.

- We call w the image of z under f.
- We call z the preimage of w under f.
- We call $D \subset \mathbb{C}$ the domain of f, and can be any set that makes sense for a given rule.
- The set E is called the co-domain of f.
- We call the set $f(D) = \{w = f(z) : z \in D\}$ of all images of D the range of f.
- We say that f is **onto** if $f(D) = E$.
- We say that f is **one-to-one** on D if $z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)$.

For each $b \in E$, we define $f^{-1}(b)$ to be the set of elements in D whose image is b. Note that $f^{-1}(b)$ may be empty if f is not onto. However, if f is one-to-one and onto, $f^{-1} : E \to D$ is also a one-to-one and onto function, called the inverse function of f.

Remarks.

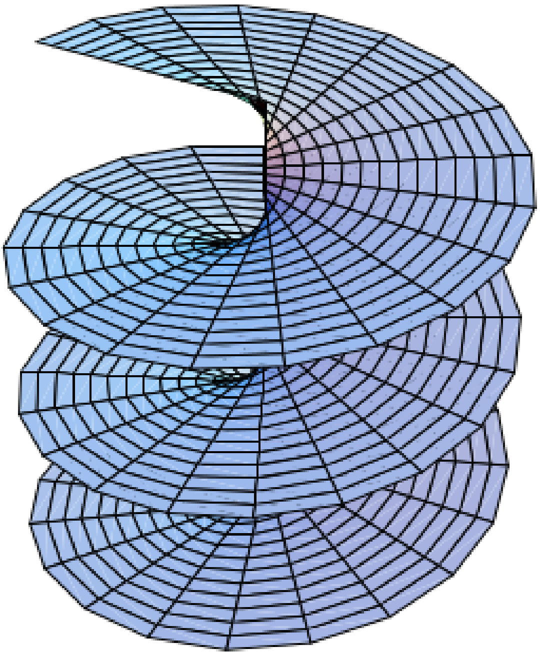


Figure 3.2: Riemann surface of $w = \arg z$

- When the context is obvious, we omit the phrase *complex-valued*, and simply refer to a function f, or to a complex function f.
- Strictly speaking, f stands for the function and $f(z)$ for the value of the function at z. However, when there is no ambiguity, we will sometimes use the time-honored notational abuse of referring to $f(z)$ as a function.
- In some contexts functions are referred to as mappings or transformations.

Examples.

(1) The function $w = f(z) = az + b, a \neq 0$, is one-to-one and onto in $\mathbb{C}$ and the inverse function is defined by $z = (w - b)/a$. Note that both are defined in the whole plane $\mathbb{C}$.

(2) The function f defined by $f(z) = z^2$ is not one-to-one because $f(i) = f(-i) = -1$. However if we restrict the domain to Re $z > 0$ it would be one-to-one.

(3) The function f defined by $w = f(z) = \arg z$ is infinite valued as for each z we have infinitely may representations of $\arg z$. We have in figure 3.2 the Riemann surface representation of $w = \arg z$.

3.1 Extending Functions from $\mathbb{R}$ to $\mathbb{C}$

We are about to extend real valued functions such as e^x from a function defined on $\mathbb{R}$ to a function defined on $\mathbb{C}$. It is reasonable to expect that this is possible in many ways, and that our extensions are chosen for "historical reasons only". Amazingly, this is false. It turns out that there is at most one way to extend a function defined on a subset of $\mathbb{R}$ to a

holomorphic extension over the complex numbers. This is due a very important principle of complex analysis called **analytic continuation**.
Most standard functions of calculus satisfy certain identities. For example for every real number x we have $\sin^2 x + \cos^2 x = 1$. Amazingly, these identities remain true over the complex numbers, and even more amazingly, there is an abstract theorem which proves this without bothering to check any special case.
Here are the two theorems in question, known as the **identity theorems**.

Identity Theorem 1. Suppose D is a domain in $\mathbb{C}$ such that $D \cap \mathbb{R} \neq \varnothing$. If f is a complex valued function defined on $D \cap \mathbb{R}$, then f can be extended to a holomorphic function on D in at most one way.

Identity Theorem 2. Suppose D is a domain in $\mathbb{C}$ such that $D \cap \mathbb{R} \neq \varnothing$. Suppose $f(z)$ and $g(z)$ are analytic on D and $f(x)$ and $g(x)$ satisfy an algebraic identity on $D \cap \mathbb{R}$. Then $f(z)$ and $g(z)$ satisfy the same identity on all of D.

3.2 The Square and Square Root Functions

The Function $w = z^2$

Let us consider the the square function $w = z^2$.

$$w = u + iv = (x^2 - y^2) + i(2xy)$$

This function maps the point (a, a) in the z plane onto the point $(0, 2a^2)$ in the w plane. That is, the ray $y = x$, with $x > 0$, is mapped onto the ray $(0, v)$, with $v > 0$; and the ray $y = x, x < 0$, is also mapped onto the ray $(0, v), v > 0$. In other words, the line $y = x$ is twice mapped onto the ray $(0, v), v \geq 0$ (see Figure). Observe that the function $w = z^2$ is not one-to-one.

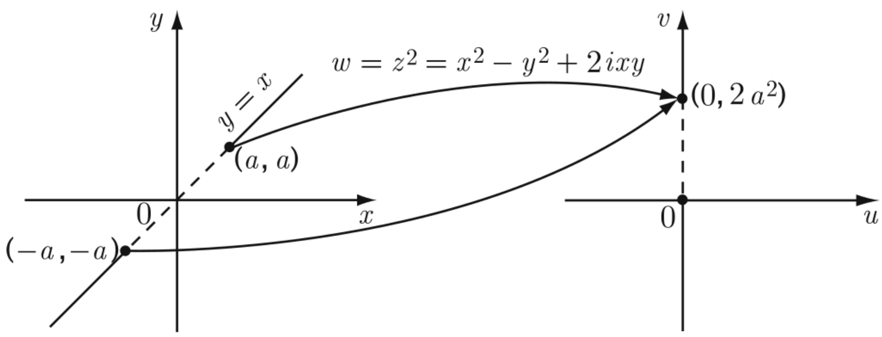


Figure 3.3: Image of the line $y = x$ under $w = z^2$

In general the point (x, mx) is mapped onto the point $(u, v) = ((1 - m^2)x^2, 2mx^2)$.
Since

$$\frac{v}{u} = \frac{2m}{1 - m^2} (m \neq \pm 1),$$

the straight line $y = mx$ is mapped twice onto the ray

$$v = \frac{2m}{1-m^2}u,$$

where u assumes all the nonnegative real numbers if $|m| < 1$ and all nonpositive real numbers if $|m| > 1$.

If we write $z = re^{i\theta}$ then

$$w = f(z) = z^2 = r^2 e^{i2\theta},$$

thus we have

$$|w| = r^2 = |z|^2 \quad \text{and} \quad \arg w = 2\theta = 2\arg z.$$

Thus a point with polar coordinates (r, θ) in the z plane is mapped onto the point with polar coordinates $(r^2, 2\theta)$ in the w plane, a point whose distance from the origin is squared and whose argument is doubled.

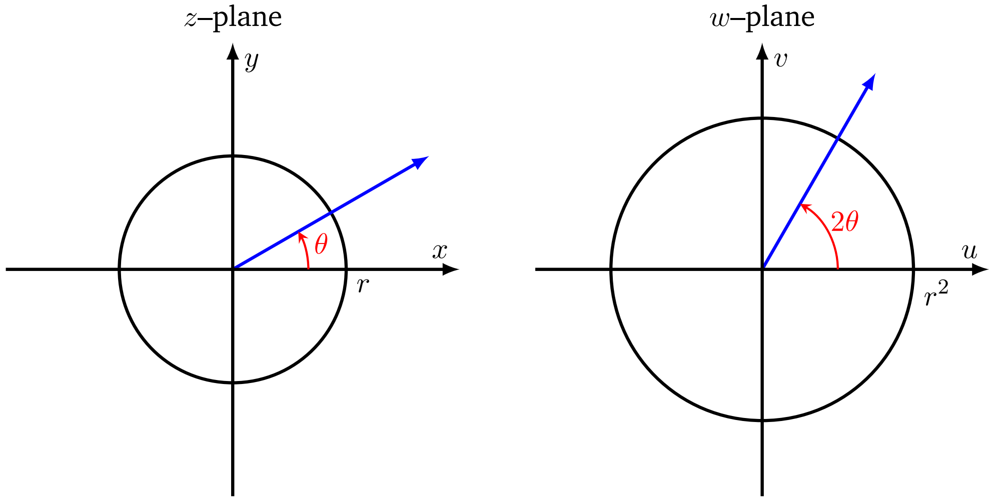


Figure 3.4: The square function $w = f(z) = z^2$.

Using the above equations we can show that:

- The circle $|z| = r_0$ in the z–plane is mapped to the circle $|w| = r_0^2$ in the w–plane. Furthermore, as z makes one complete loop, the image w makes two complete loops.
- A ray $\arg z = \theta_0$ from the origin in the z-plane is mapped to a ray in the w-plane of twice the angle. As z traverses the ray from the origin to ∞ constant speed, the value w traverses the image ray from 0 to ∞, starting slowly and increasing its speed.
- The positive real axis in the z-plane, which is a ray with angle 0, is mapped to the positive real axis in the w-plane.
- The right half-plane $\operatorname{Re} z > 0$ is mapped onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$.
- The sector $|\operatorname{Arg}(z)| < \theta_0 \in (0, \pi/2]$ is mapped onto the sector $|\operatorname{Arg}(w)| < 2\theta_0$.
- The upper half $z-$plane including the real axis, $\operatorname{Im} z \geq 0$, is mapped to the entire

$w-$plane.

- Any semi circle of radius r centered at the origin is mapped onto circle of radius r^2 centered at the origin.

Let us compare the function $w = z^2$ with its real-valued counterpart, the parabola $y = x^2$. The line $y = c$ in the z plane is transformed into $u = x^2 - c^2$ and $v = 2xc$, from which we obtain

$$u = \frac{v^2}{4c^2} - c^2.$$

Hence the horizontal line $y = c \neq 0$ is mapped onto the parabola

$$u = \frac{v^2}{4c^2} - c^2.$$

If $c = 0$, the parabola degenerates into the ray $(u, 0), u \geq 0$.

In a similar fashion, we can show that the vertical line $x = a \neq 0$ maps onto the parabola (see Figure)

$$u = -\frac{v^2}{4c^2} + a^2.$$

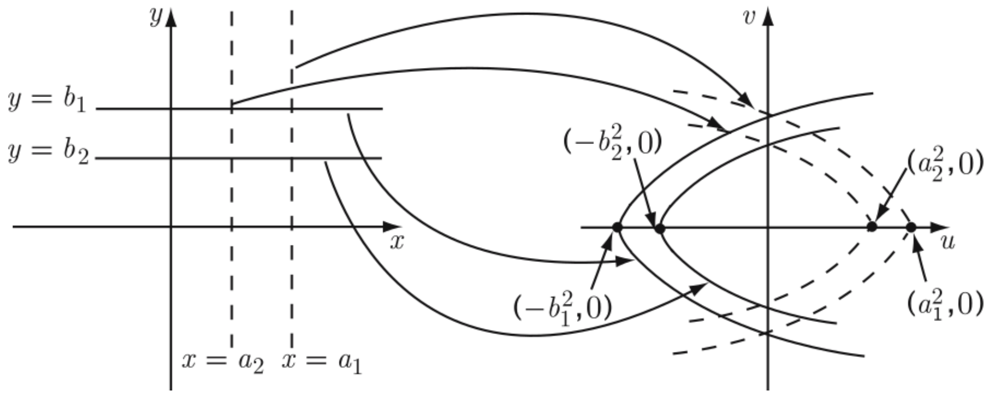


Figure 3.5: Image of lines parallel to coordinate axes under $w = z^2$

Example. Determine the region of the w plane into which each of the following is mapped by the transformation $w = z^2$.

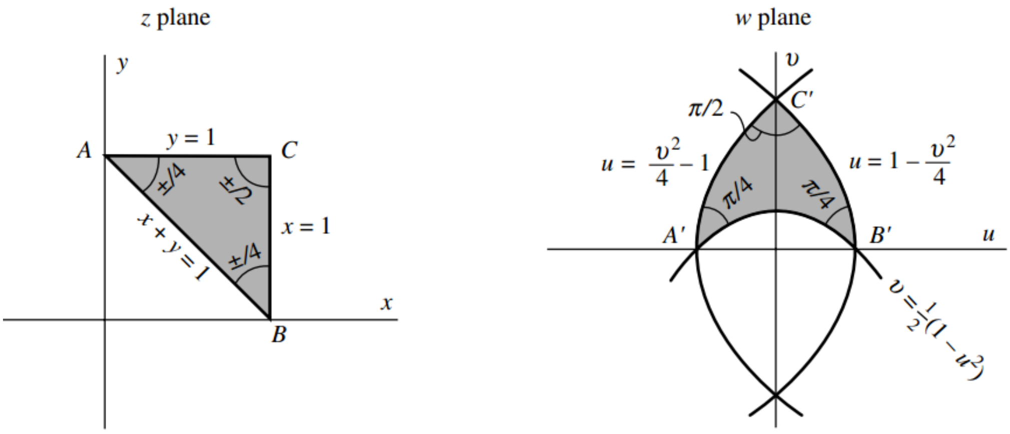
(a) First quadrant of the $z-$plane.

(b) Region bounded by $x = 1, y = 1$ and $x + y = 1$.

Solution.

(a) We would like to find the image of $A = \{re^{i\theta} : 0 \leq \theta \leq \pi/2\}$ under the map $w = z^2$. Thus $f(A) = \{w : |w| = r^2 \text{and} \arg(w) = 2\theta \in (0, \pi)\}$.

(b) Since $w = u + iv = (x + iy)^2 = (x^2 - y^2) + i(2xy)$ then
$x = 1$ maps onto $u = 1 - y^2, v = 2y \Rightarrow u = 1 - v^2/4$;
$y = 1$ maps onto $u = x^2 - 1, v = 2x \Rightarrow u = v^2/4 - 1$ and
$x + y = 1$ maps onto $u = x^2 - (1 - x)^2, v = 2x(1 - x) \Rightarrow v = (1 - u^2)/2$.



The function $w = z^{1/2}$

Now we turn to the problem of finding an inverse function for $w = z^2$. Every point $w \neq 0$ is hit by exactly two values of z, the two square roots $\pm\sqrt{w}$. In order to define an inverse function, we must restrict the domain in the z-plane so that values w are hit by only one value of z in the z–plane.
Note if $z = re^{i\theta}$ where $\theta \in (-\pi/2, \pi/2)$, then $\arg w \in (-\pi, \pi)$.
This leads us to draw a slit, or branch cut, in the w-plane along the negative axis from $(-\infty, 0]$, and to define the inverse function on the slit plane $\mathbb{C} \setminus (-\infty, 0]$. Every value w in the slit plane is the image of exactly two z-values, one in the open right half-plane $\operatorname{Re} z > 0$, the other in the left half-plane $\operatorname{Re} z < 0$. Thus there are two possibilities for defining a (continuous) inverse function on the slit plane which make the square root a 2–valued function. We refer to each determination of the inverse function as a branch of the inverse. One branch $f_1(w)$ of the inverse function is defined by declaring that $f_1(w)$ is the value z such that $\operatorname{Re} z > 0$ and $z^2 = w$. Then $f_1(w)$ maps the slit plane $\mathbb{C} \setminus (-\infty, 0]$ onto the right half-plane $\operatorname{Re} z > 0$, and it forms an inverse for z^2 on that half-plane.
The function $f_1 : \mathbb{C} \setminus (-\infty, 0] \to \{\operatorname{Re} z > 0\}$ is called the **principal branch** of $w^{1/2}$ and is expressed as

$$f_1(w) = |w|^{1/2} e^{i \operatorname{Arg} w/2} = \sqrt{w}, \quad w \in \mathbb{C} \setminus (-\infty, 0].$$

The other branch of $w^{1/2}$ is $f_2 : \mathbb{C} \setminus (-\infty, 0] \to \{\operatorname{Re} z < 0\}$ is defined as

$$f_2(w) = |w|^{1/2} e^{i \operatorname{Arg} w/2 + i\pi} = -f_1(w) = -\sqrt{w},$$

and maps the slit plane $\mathbb{C} \setminus (-\infty, 0]$ onto the open left half-plane $\operatorname{Re} z < 0$.

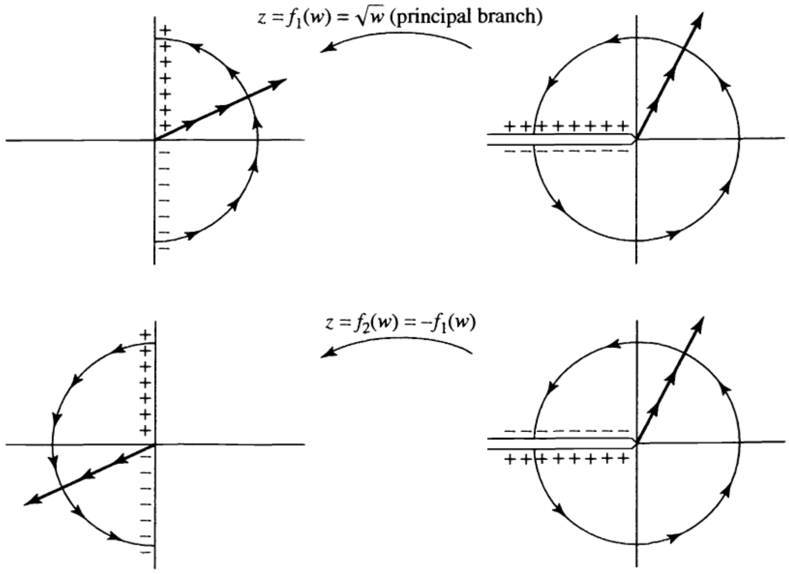


Figure 3.6: Branches of the square root function.

Notation. If x is a positive real number the $x^{1/2} = \sqrt{x}$. However if $z \in \mathbb{C}$ then $z^{1/2} = \pm\sqrt{z}$ i.e. both branches of branches of the square root function.

3.3 The Exponential Function $w = e^z$

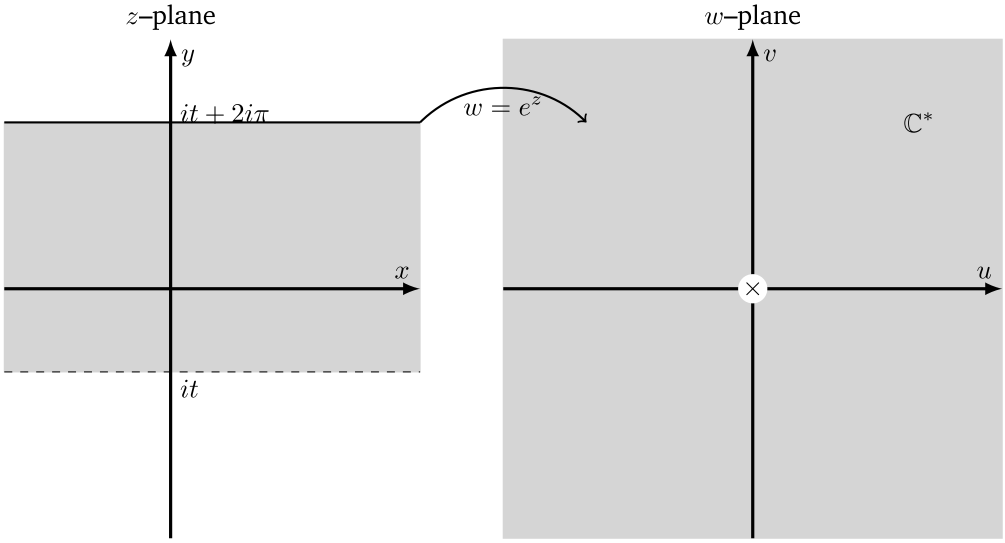
If $z = x + iy \in \mathbb{C}$, then the exponential complex valued function is defined as:

$$\boxed{w = e^z = e^{x+iy} = e^x(\cos y + i \sin y) = u + iv}$$

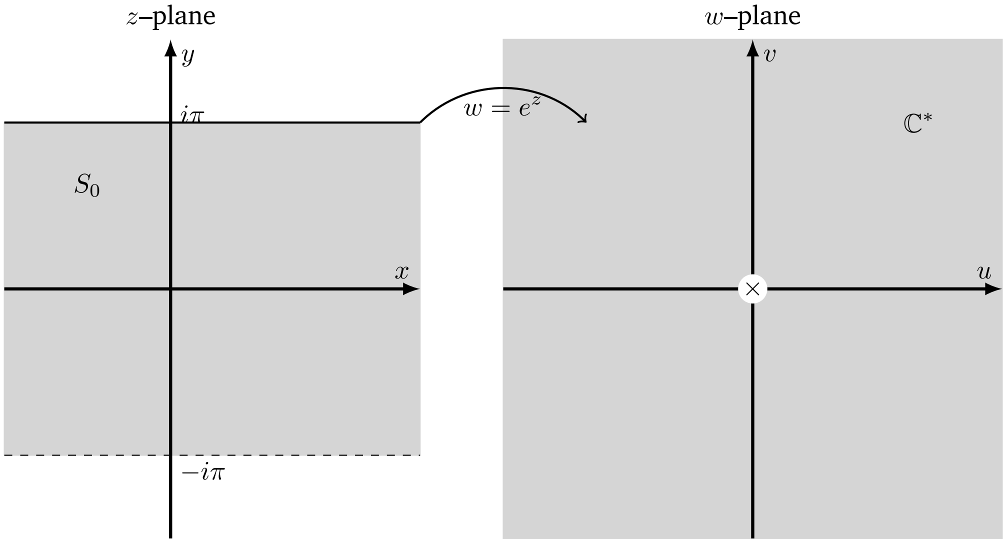
where e is the natural base of logarithms. Note that when $y = 0$, the right hand side is simply the real function e^x. So our definition extends the usual real valued exponential function. We can conclude from the definition of $e^z = e^{x+iy} = e^x e^{iy}$ that

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2k\pi.$$

Note that since $|e^z| = e^x > 0$, then $e^z \neq 0$. Furthermore $e^{z+2i\pi} = e^z e^{2i\pi} = e^z$, this means that e^z is periodic function with period $2i\pi$. We can easily check using the Cauchy-Riemann equation that e^z is an entire function and that its derivative $(e^z)' = e^z$. The image of any horizontal strip of width 2π is $\mathbb{C}^* = \mathbb{C} - \{0\}$.



The special case map $\exp : S_0 \to \mathbb{C}^*$ is one-to-one and onto on S_0 and admits an inverse.



Remark.

$$\boxed{e^{i\pi} + 1 = 0}$$

is **the most beautiful equation** in all of mathematics. It contains the five most important constants as well as the three most important operations (addition, multiplication and exponentiation).

Properties: For $z = x + iy, z_1, z_2 \in \mathbb{C}$, the following assertions are true:

(1) $e^0 = e^0(\cos 0 + i \sin 0) = 1$.

(2) $e^z \neq 0$ for all z in $\mathbb{C}$.

(3) $e^z = 1 \Leftrightarrow z = 2ik\pi; k \in \mathbb{Z}$.

(4) $e^{-z} = 1/e^z$.

(5) $|e^z| = e^x = e^{\operatorname{Re} z}$ and $\arg(e^z) = y + 2k\pi$.

(6) $e^{(z+2\pi i)} = e^z$ which shows that e^z is periodic with period $2i\pi$.
(7) $e^{z_1+z_2} = e^{z_1}e^{z_2}$.
(8) $\lim_{z\to z_1} e^z = e^{z_1}$.
(9) $(e^z)' = e^z$.

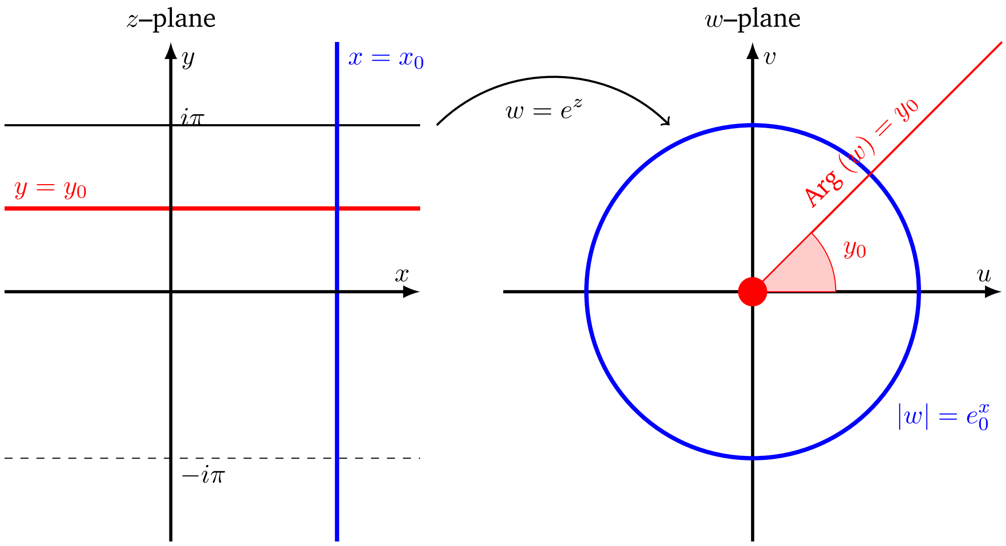


Figure 3.7: Images of a vertical and horizontal line via the mapping $w = e^z$.

Mapping Properties of $w = e^z$.

Since $e^{z+2ik\pi} = e^z$ for every $k \in \mathbb{Z}$ then the points $x_0 + i(y_0 + 2k\pi)$ have the same image for every integer k. Hence we may examine the mapping properties by restricting ourselves to the infinite strip $-\pi < \text{Im}\, z \le \pi$. Whatever occurs in this strip will also occur in the strip $-\pi + 2k\pi < \text{Im}\, z \le \pi + 2k\pi$. With this restriction, $\text{Arg}(w) \in (-\pi, \pi]$. We have the following:

- The line segment $x = x_0, -\pi < y \le \pi$, is mapped one-to-one onto the circle in the $w-$plane having center at the origin and radius e^{x_0}.

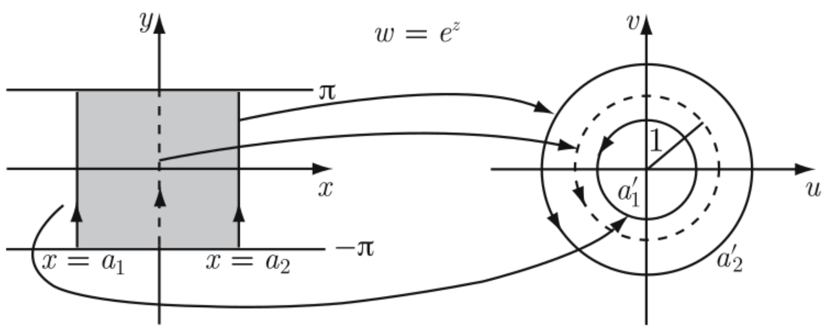


Image of line segments parallel to coordinate $y-$axes

- Since $|e^z| = e^x > 1$ if and only if $x > 0$, the semi-infinite-strip

$$\{z : \text{Re}\, z > 0, -\pi < \text{Im}\, z \leq \pi\}$$

is mapped one-to-one onto $\{w : |w| > 1\}$, while the strip

$$\{z : \text{Re}\, z < 0, -\pi < \text{Im}\, z \leq \pi\}$$

is mapped onto the punctured unit disc $\{w : 0 < |w| < 1\}$.

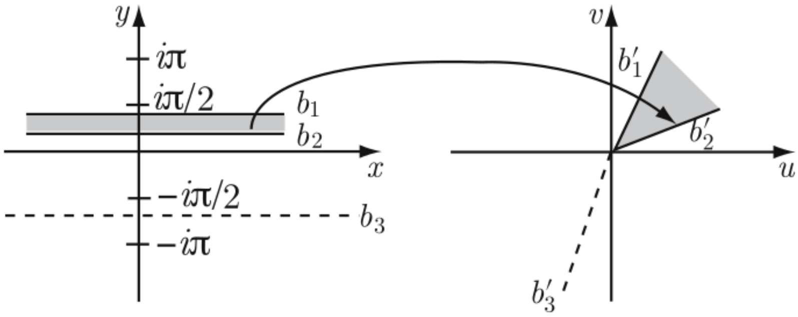


Image of line segments parallel to coordinate $y-$axes

- As $|e^z| = e^x < 1$ if and only if $x < 0$, the semi-infinite strip

$$\{z : \text{Re}\, z < 0, 0 \leq \text{Im}\, z \leq \pi\}$$

is mapped one-to-one onto the upper semi-disc

$$\{w : \text{Im}\, w \geq 0, |w| < 1\}$$

excluding the origin.
- Since e^x describes the positive reals, the line $y = y_0$ is mapped one-to-one onto the ray $\text{Arg}\, w = y_0$. Therefore, the infinite strip

$$\{z : 0 < \text{Im}\, z < \pi\}$$

is mapped one-to-one onto the upper half-plane $\{w : \text{Im}\, w > 0\}$, while the strip

$$\{z : -\pi < \text{Im}\, z < 0\}$$

is mapped onto the lower half-plane $\{w : \text{Im}\, w < 0\}$.
- Note that the $x-$axis, $y = 0$, is mapped onto the positive real axis and the line $y = \pi$ is mapped onto the negative real axis.
Hence, under the exponential mapping $w = e^z$, the strip

$$\{z : -\pi < \text{Im}\, z \leq \pi\}$$

is mapped one-to-one onto the punctured $w-$plane, $\mathbb{C}^*$.

- We can combine the previous mappings to determine the image of the rectangles under the mapping $w = e^z$. Writing the image in the polar form, we have the rectangle
$$\{z : A \le x \le B, -\pi < C \le y \le D \le \pi\}$$
being mapped onto the region
$$\{Re^{i\theta} : e^A \le R \le e^B, C \le \theta \le D\},$$
bounded by arcs and rays.

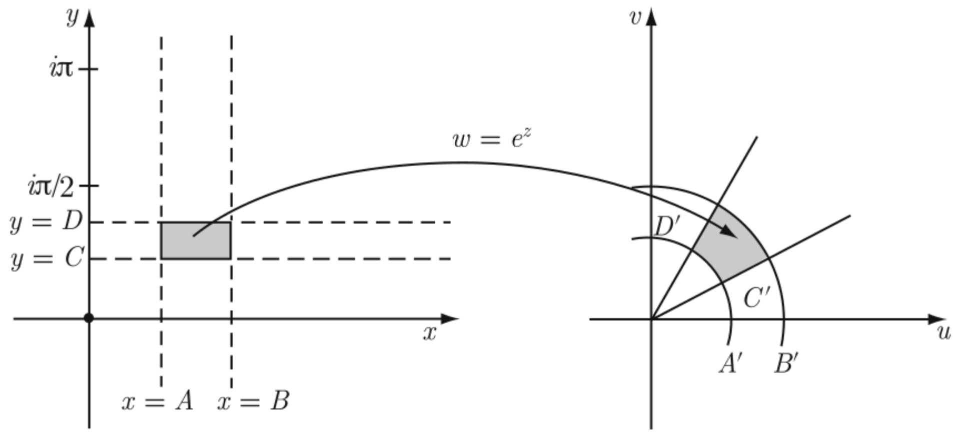
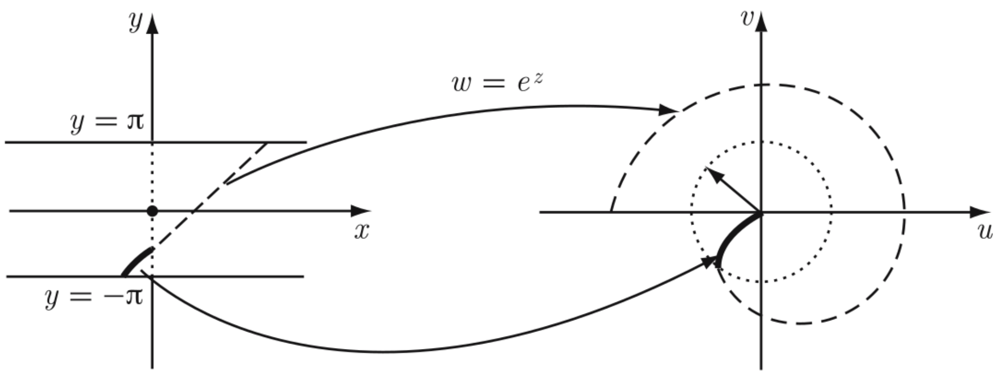


Image of a rectangle under e^z

Next consider a straight line not parallel to either of the coordinate axes. The image of this line will have neither constant modulus nor constant argument, yet it must grow arbitrarily large as x grows arbitrarily large, and must make a complete revolution each time y increases by 2π, thus producing a spiraling effect.

If $y = mx + b, m \neq 0$, then
$$w = e^z = e^{x+i(mx+b)}.$$
Hence $|w| = |e^z| = e^x$ and $\text{Arg}(w) = mx + b + 2k\pi$, where $k \in \mathbb{Z}$ is chosen such that $\text{Arg}(w) \in (-\pi, \pi]$. Eliminating x from the relations we get
$$|w| = e^{(\theta-b-2k\pi)/m} = Ke^{\alpha/m},$$
where $K > 0, \alpha = \theta - 2k\pi \in \mathbb{R}$ and the above equation represents what is known as a logarithmic spiral.



Logarithmic spiral

3.4 The logarithm function $w = \log z$

For $z \neq 0$, we define $\log z$ to be the multiple-valued function

$$w = \log(z) = \ln|z| + i\arg(z) = \ln|z| + i\,\mathrm{Arg}\,z + i2k\pi,\ k \in \mathbb{Z}$$

which has infinitely many values at each point $z \neq 0$, carried from the multi-valudeness of $\arg z$. The values of $\log z$ are precisely the complex numbers w such that $e^w = z$.

$$e^w = e^{\log z} = e^{\ln|z|}e^{i\,\mathrm{Arg}\,z}e^{i2k\pi} = |z|e^{i\,\mathrm{Arg}\,z} = z.$$

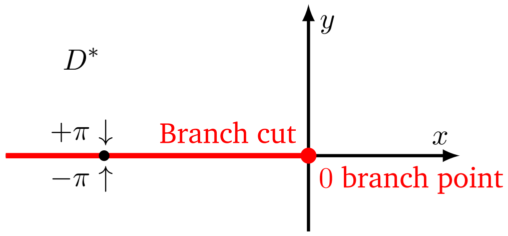
Note again that $\log(e^z) \neq z$.

$$\log e^z = \ln|e^z| + i\arg e^z = \ln|e^x| + i\arg e^{x+iy} = x + i(y + 2k\pi) = z + i2k\pi \neq z.$$

As we have done for the $\arg z$, it is appropriate to define the single valued principal branch of the logarithm by choosing the principal argument of z that satisfies $-\pi < \mathrm{Arg}\,z \leq \pi$. For $z \neq 0$, we call the **principal branch** or **principal value** of $\log z$ and denote it by $\mathrm{Log}\,z$ the complex valued function defined by :

$$\mathrm{Log}\,z := \ln|z| + i\,\mathrm{Arg}\,z;\ -\pi < \mathrm{Arg}\,z \leq \pi$$

Thus $\mathrm{Log}\,z$ is a single-valued inverse for e^w. The restriction in the above equation may be viewed geometrically as a cut of the $z-$plane along the negative real axis. This ray is then called the **branch cut** for the function $\mathrm{Log}\,z$. Note that for positive, real numbers the principal branch, $\mathrm{Log}\,x = \ln x$ is real-valued which makes it the extension of the natural logarithm function from $\mathbb{R}$ to $\mathbb{C}$.



Consequently we also have

$$w = \log(z) = \ln|z| + i\arg(z) = \ln|z| + i\,\mathrm{Arg}\,z + i2k\pi = \mathrm{Log}\,z + +i2k\pi,\ k \in \mathbb{Z}.$$

Example Here are a few evaluations of $\mathrm{Log}\,z$:

$$\begin{aligned}
\mathrm{Log}(2) &= \ln|2| + i\,\mathrm{Arg}(2) = \ln(2)\\
\mathrm{Log}(i) &= \ln|i| + i\,\mathrm{Arg}(i) = \ln(1) + i\pi/2 = i\pi/2\\
\mathrm{Log}(-2) &= \ln|-2| + i\,\mathrm{Arg}(-2) = \ln(2) + i\pi\\
\mathrm{Log}(1-i\sqrt{3}) &= \ln|1-i\sqrt{3}| + i\,\mathrm{Arg}(1-i\sqrt{3}) = \ln(2) - i\pi/3
\end{aligned}$$

Using the above information we can then write the multi-valued logarithm function as

Once we know the principal value $\mathrm{Log}\,z$, we obtain all values of $\log z$ by simply adding $i2k\pi$. Here are evaluations of $\log z$:

$$\begin{aligned}
\log(2) &= \ln|2| + i\,\mathrm{Arg}(2) + i2k\pi = \ln(2) + i2k\pi\\
\log(i) &= \ln|i| + i\,\mathrm{Arg}(i)i2k\pi = \ln(1) + i\pi/2 = i\pi/2 + i2k\pi\\
\log(-2) &= \ln|-2| + i\,\mathrm{Arg}(-2)i2k\pi = \ln(2) + i\pi + i2k\pi\\
\log(1-i\sqrt{3}) &= \ln|1-i\sqrt{3}| + i\,\mathrm{Arg}(1-i\sqrt{3}) + i2k\pi = \ln(2) - i\pi/3 + i2k\pi
\end{aligned}$$

Example. Find the values of $\mathrm{Log}(e), \mathrm{Log}(-e), \mathrm{Log}(1)$ and $\mathrm{Log}(-1)$.

- $\mathrm{Log}(e) = \ln|-e| + i\,\mathrm{Arg}(e) = \ln(e) = 1$
- $\mathrm{Log}(-e) = \ln|-e| + i\,\mathrm{Arg}(-e) = 1 + i\pi$
- $\mathrm{Log}(1) = \ln|1| + i\,\mathrm{Arg}(1) = \ln(1) = 0$
- $\mathrm{Log}(-1) = \ln|-1| + i\,\mathrm{Arg}(-1) = i\pi$

In general if $x > 0$ we have

- $\mathrm{Log}(x) = \ln|x| + i\,\mathrm{Arg}(x) = \ln(x)$
- $\mathrm{Log}(-x) = \ln|-x| + i\,\mathrm{Arg}(-x) = \ln(x) + i\pi$

This confirms that indeed $\mathrm{Log}(z)$ is the extension of $\ln(x)$ from $\mathbb{R}$ to $\mathbb{C}$.

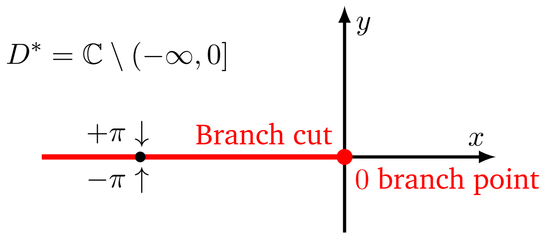
Domain of Analyticity of $\operatorname{Log} z$

A **branch** of $\log z$ is any single-valued function $F(z)$ that satisfies the identity $e^{F(z)} = z$ for all nonzero complex values of z. There are infinitely many branches associated with the multiple-valued function $\log z$. Each is an inverse of the function e^z. Among all the branches for $\log z$, there is exactly one whose imaginary part $(\arg z)$ is defined in the interval $(-\pi, \pi]$. This branch is called the **principal branch** of $\log z$ and is denoted as

$$\operatorname{Log} z := \ln|z| + i\operatorname{Arg} z; \ -\pi < \operatorname{Arg} z \leq \pi \ .$$

How do we choose a domain D^* in which $\operatorname{Log} z$ would be analytic?
Note that $\operatorname{Arg} z$ is not continuous on $(-\infty, 0]$ and hence $\operatorname{Log} z$ is not analytic on $(-\infty, 0]$, which makes it a set on non-isolated singularities. In order to make $w = \operatorname{Log} z$ analytic its domain of analyticity must be the slit $z-$plane $D^* = \mathbb{C} \setminus (-\infty, 0]$. Furthermore

$$\frac{d}{dz}\operatorname{Log} z := \frac{1}{z} \text{ for } \ z \in D^*$$



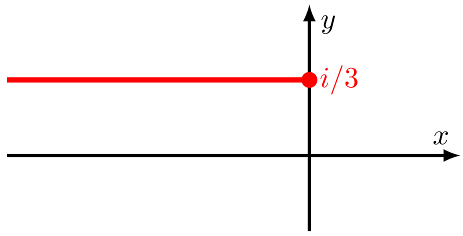
Remark.
We have seen that $\operatorname{Log} z$ is not s not continuous on the negative real axis. This does not mean that the logarithm function is not continuous on the negative real axis. All we have seen is that $\operatorname{Log} z$, the principal branch, is not continuous at these points. By making our cut along a different ray, we can find a branch of the logarithm that is continuous for negative real values. For instance, the single-valued function

$$w = \log_{-\frac{\pi}{2}}(z) = \ln|z| + i\arg z; \ (-\pi/2 < \arg z \leq 3\pi/2)$$

is continuous at all points on the negative real axis, but not on the ray $\arg z = -\pi/2$. In other words, the logarithm function is continuous for all nonzero complex values in the following sense: Given $z_0 \neq 0$, there exists a branch for which $\lim_{z \to z_0} \log z = \log z_0$.
However, there does not exist a branch for which $\log z$ is continuous for all nonzero complex numbers.

Example. Determine the domain of analyticity of $w = \operatorname{Log}(3z - i)$.
This function is analytic by the chain rule in $\mathbb{C}$ except where
$\operatorname{Re}(3z) = 3x - 1 \leq 0 \Rightarrow x \leq 0$ and $\operatorname{Im}(3z - i) = 3y - 1 = 0 \Rightarrow y = 1/3$.

Hence the domain of analyticity is $\mathbb{C} \setminus (-\infty, i/3]$.



Mapping Properties of $w = \operatorname{Log} z$.

Since the exponential function maps horizontal lines to rays issuing from the origin and maps vertical lines to circles , its inverse, the logarithm function, maps rays issuing from the origin to horizontal lines and circles to vertical lines . In fact, the ray $\operatorname{Arg} z = \theta_0$ is mapped onto the horizontal line $\Im w = \theta_0$. As z traverses the ray from 0 to ∞, the image w traverses the entire horizontal line from left to right. As θ_0 increases between $-\pi$ and π, the rays sweep out the slit plane $\mathbb{C} \setminus (-\infty, 0]$, and the image lines fill out a horizontal strip $-\pi < \operatorname{Im} w < \pi$ in the $w-$plane. Similarly the image of $\{|z| = r, -\pi < \arg z < \pi\}$ is the vertical vertical segment $\{\operatorname{Re}(w) = \ln r, -\pi < \operatorname{Im} w < \pi\}$.
Since

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z; -\pi < \operatorname{Arg} z < \pi,$$

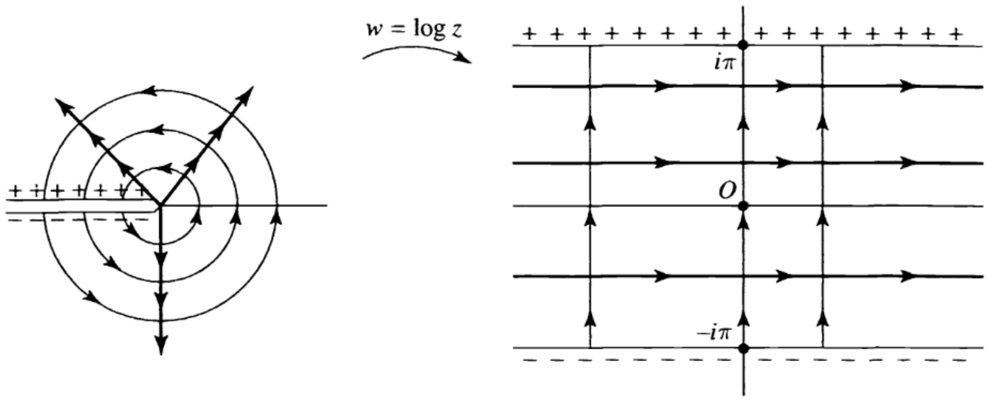


Image of slit plane $\mathbb{C} \setminus (-\infty, 0]$ under under $w = \operatorname{Log} z$

- The image of the circle $|z| = r$ for the function $w = \operatorname{Log} z$ is the line segment $u = \ln r, \pi < v \leq \pi$

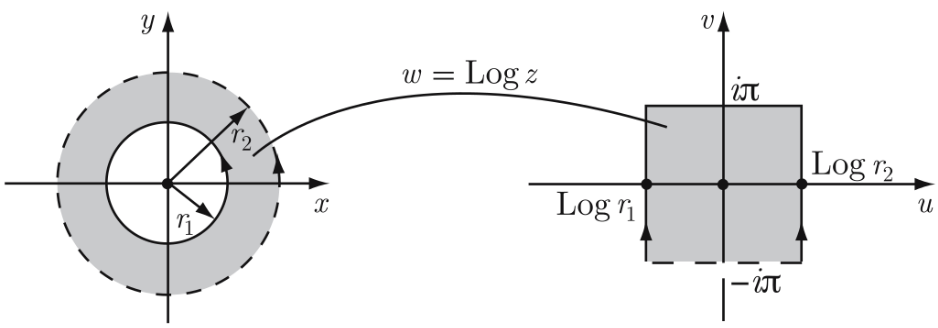


Image of an annulus region under $w = \operatorname{Log} z$

- The ray $\operatorname{Arg} z = \theta$ is mapped onto the line $v = \theta$.

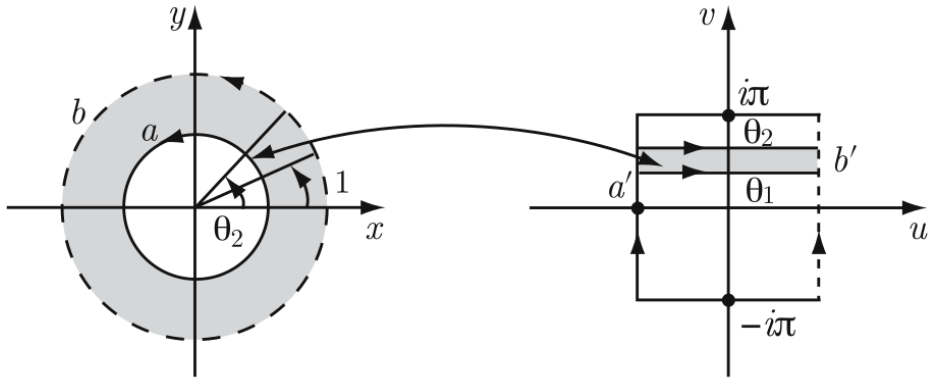


Image of segment of rays under under $w = \operatorname{Log} z$

Example. Find the image of the annulus $e \le |z| \le e^5$ via the function $w = \operatorname{Log} z$.
Solution. Since $w = \ln|z| + i \operatorname{Arg} z$ then $u = \ln|z|$ and $v = \operatorname{Arg} z$.
Furthermore $e \le |z| \le e^5$, then $\ln e = 1 \le u = \ln|z| \le \ln e^5 = 5$ and $-\pi < v = \operatorname{Arg} z \le \pi$.

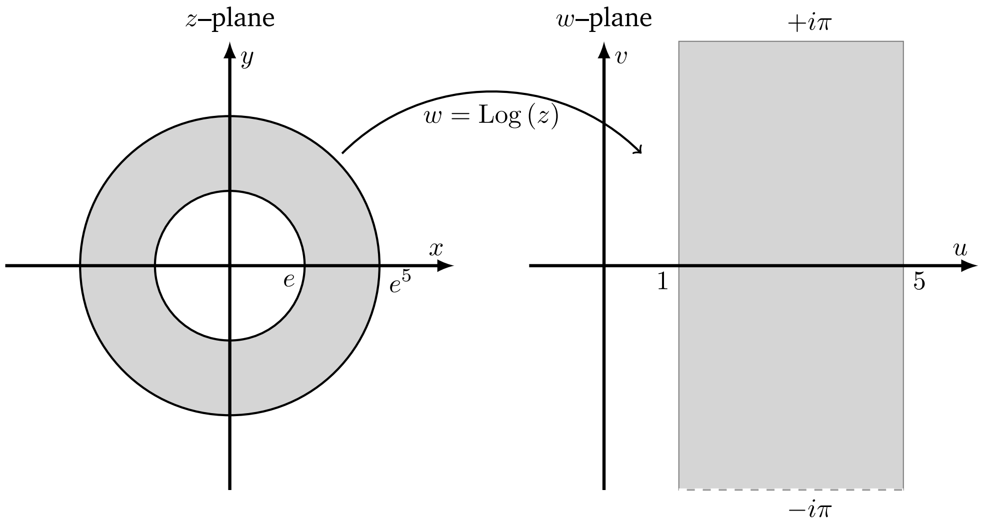


Figure 3.8: Mapping of the annulus via $w = \operatorname{Log} z$.

3.5 The complex exponent function $w = z^a$

Consider z^a where z and a are complex numbers and z is nonzero. We define this expression in terms of the exponential and the logarithm as

$$z^a = e^{a \log z} = e^{a(\operatorname{Log} z + 2k\pi i)} = e^{a \operatorname{Log} z} e^{2ak\pi i};\ k \in \mathbb{Z}.$$

The first factor $e^{a \operatorname{Log} z}$ is single valued but the second factor $A = e^{2ka\pi i}$ may be multiple-valued depending on the value of a.

Case 1. If $a \in \mathbb{Z}$, then $A = 1$ and hence z^a is single valued.
Case 2. If $a = m/n \in \mathbb{Q}$, then $A = e^{2km\pi i/n}$ and hence z^a is $n-$valued.
Case 3. If $a \notin \mathbb{Q}$ then A and hence z^a is infinite-valued.

Next suppose that $a = a_1 + ia_2$ (a_1 and a_2 real, $a_2 \neq 0$). Then

$$z^{a_1+ia_2} = e^{(a_1+ia_2)\log z} = e^{a_1 \ln r - a_2(\theta+2k\pi)} e^{i(a_2 \ln r + a_1\theta + 2k\pi a_1)}$$

Since $|z^{a_1+ia_2}| = r^{a_1} e^{-a_2(\theta+2k\pi)}$, the complex number $z^{a_1+ia_2}$ has a different modulus for each branch, any two of which differ by a factor of $e^{-2k\pi}$, k an integer.

Examples. Find all possible values of the following:
(1) $5^{1/2} = e^{(1/2)\log 5} = e^{(1/2)(\ln 5 + 2ki\pi)} = e^{(1/2)(\ln 5)} e^{ki\pi} = \pm\sqrt{5}$
(2) $i^{1/2} = e^{(1/2)\log i} = e^{(1/2)(i\pi/2+2ki\pi)} = \pm e^{i\pi/4} = \pm\frac{\sqrt{2}}{2}(1+i)$
(3) $i^i = e^{i\log i} = e^{i(\ln 1 + i\pi/2 + 2k\pi)} = e^{-(\pi/2+2k\pi)}$, where $k \in \mathbb{Z}$
(4) $1^\pi = e^{\pi\log(1)} = e^{\pi(\ln(1)+i2k\pi)} = e^{i2k\pi^2}$, where $k \in \mathbb{Z}$

Example. Find all solutions of $z^{1-i} = 4$.
If we rewrite the equation we get

$$e^{(1-i)\log z} = 4 = e^{\ln 4 + 2ki\pi}$$

so that

$$(1-i)\log z = 2\ln 2 + 2ki\pi \Rightarrow \log z = [\ln 2 - k\pi] + i[\ln 2 + k\pi].$$

So by the definition of $\log z$, we get

$$z = e^{[\ln 2 - k\pi] + i[\ln 2 + k\pi]} = 2e^{-k\pi} e^{i[\ln 2 + k\pi]}, k \in \mathbb{Z}.$$

Example. Consider 1^π. We apply the definition $a^b = e^{b\log a}$ to get

$$1^\pi = e^{\pi\log(1)} = e^{\pi[\ln(1)+i2k\pi]} = e^{i2k\pi^2}$$

Thus we see that 1^π has an infinite number of values, all of which lie on the unit circle $|z| = 1$ in the complex plane. However, the set 1^π is not equal to the set $|z| = 1$. There are points in the latter which are not in the former. This is analogous to the fact that the rational numbers are dense in the real numbers, but are a subset of the real numbers.

Example Consider the harmless looking equation, $i^z = 1$.
Before we start with the algebra, note that the right side of the equation is a single number. i^z is single-valued only when z is an integer. Thus we know that if there are solutions for z, they are integers. We now proceed to solve the equation.

$$i^z = 1 \Leftrightarrow \left(e^{i\pi/2}\right)^z = 1.$$

Use the fact that z is an integer to get

$$e^{iz\pi/2} = 1 = e^{i2k\pi} \Rightarrow \boxed{z = 4k; k \in \mathbb{Z}.}$$

Now let's consider a slightly different problem: $1 \in i^z$. For what values of z does i^z have 1 as one of its values.

$$1 \in i^z = e^{z\log i} \Leftrightarrow 1 \in \{e^{zi(\pi/2+2k\pi)}\} \Leftrightarrow zi(\pi/2 + 2k\pi) = i2\pi n,\ n, k \in \mathbb{Z}$$

$$\boxed{z = \frac{4n}{1+4k};\ n, k \in \mathbb{Z}}$$

There are an infinite set of rational numbers for which i^z has 1 as one of its values.
For example,

$$i^{4/5} = \{1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}\}$$

Warning !

Suppose $z = re^{i\theta}, r \neq 0,\ k, n, m \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{R}$ then :

1. $z^\alpha z^\beta = e^{\alpha \log z} e^{\beta \log z} = e^{\alpha(\ln r + i\theta + 2ik\pi)} e^{\beta(\ln r + i\theta + 2in\pi)} = e^{(\alpha+\beta)\ln r} e^{i(\alpha+\beta)\theta} e^{2i\pi(k\alpha+n\beta)}$
2. $z^{\alpha+\beta} = e^{(\alpha+\beta)\log z} = e^{(\alpha+\beta)(\log r + i\theta + 2im\pi)} = e^{(\alpha+\beta)\ln r} e^{i(\alpha+\beta)\theta} e^{2i\pi m(\alpha+\beta)}$

Clearly if $\alpha, \beta \in \mathbb{Z}$ then $z^\alpha z^\beta = z^{\alpha+\beta}$.
If either α or β is an integer, then $z^\alpha z^\beta$ and $z^{\alpha+\beta}$ assume the same set of values, although equality for each α and β need not hold. In general, $z^{\alpha+\beta}$ assumes every value of $z^\alpha z^\beta$, but the converse is not true. We have $2^{1/2+1/2} = 2$ but $2^{1/2}2^{1/2} = \pm 2$.
We leave it for the reader to show this containment for α and β complex numbers.

Square root revisited

If we use the definition of z^α then we have

$$\begin{aligned} w = z^{1/2} &= e^{(1/2)\log z} \\ &= e^{(1/2)(\operatorname{Log} z + 2ik\pi)} \\ &= e^{(1/2)(\operatorname{Log} z + 2ik\pi)} \\ &= e^{(1/2)(\ln|z| + i\operatorname{Arg} z)} e^{ik\pi} \\ &= \sqrt{|z|} e^{i[(\operatorname{Arg} z)/2 + k\pi]} \end{aligned}$$

we will get two branches of the square root for the the values of $k = 1, 2$.

$$w_2 = w_0 = z^{1/2} = \sqrt{|z|} e^{i(\operatorname{Arg} z)/2} = \sqrt{z}$$

w_2 is the principal branch and we have the second branch

$$w_1 = z^{1/2} = \sqrt{|z|} e^{i(\operatorname{Arg} z)/2 + i\pi} = -w_2 = -\sqrt{z}$$

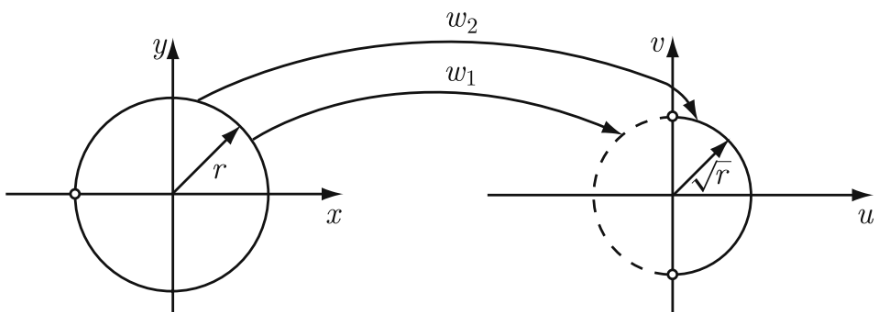


Figure 3.9: Mapping properties of square root function

Both w_1 and w_2 are continuous functions, except on the negative real axis. This ray is called a branch cut for both w_1 and w_2. Each of these single-valued functions is called a determination or branch of the multiple-valued function $w = z^{1/2}$.
We now establish some mapping properties for the functions w_1 and w_2. The punctured plane $(z \neq 0)$ is mapped by w_2 onto the right half-plane, including the positive imaginary axis, and by w_1 onto the left half-plane, including the negative imaginary axis. These functions also map circles onto semicircles, excluding the end point (see Figure).

Useful Identities and Inequalities.

The complex logarithm obeys many of the algebraic identities that we expect from the real logarithm, only that we have to take into account its multiple-valuedness properly. Therefore an identity like

$$\boxed{\log(ab) = \log a + \log b,}$$

for nonzero complex numbers a and b , is still valid in the sense that having chosen a value (out of the infinitely many possible values) for $\log(a)$ and for $\log(b)$, then there is a value of $\log(ab)$ for which the above equation holds.
In the complex plane we have for $z_1, z_2 \in \mathbb{C}^*$;

$$\boxed{\log(z_1 z_2) = \log(z_1) + \log(z_2)}$$

however the equality is interpreted as a set equality. This means for any value of $\log(z_1 z_2)$ can be expressed as the sum of some value of $\log(z_1)$ and some value of $\log(z_2)$. In addition, the sum of any values of $\log(z_1)$ and $\log(z_2)$ can be expressed as some value of $\log(z_1 z_2)$. With that in mind we have the following identities and inequalities:

- $a^b = e^{b \log a}$
- $e^{\log z} = e^{\mathrm{Log}\, z}$
- $\log(ab) = \log a + \log b$
- $\log(1/a) = -\log a$
- $\log(a/b) = \log a - \log b$
- $\log(z^{1/n}) = (\log a)/n, n \in \mathbb{N}$

Warning ! The reader should verify the following:

- $\operatorname{Log}(ab) \neq \operatorname{Log} a + \operatorname{Log} b$
- $\log(z^a) \neq a \log z$
- $\operatorname{Log}(z^a) \neq a \operatorname{Log} z$
- $\log e^z \neq z$
- $\log(z^2) = \log(z) + \log(z) \neq 2\log(z)$
- $\operatorname{Log}(z_1 z_2) \neq \operatorname{Log}(z_2) + \operatorname{Log}(z_2)$

3.6 Trigonometric and Hyperbolic Functions

Trigonometric Functions

Just as we extended the real exponential function, we now extend the familiar real trigonometric functions to complex trigonometric functions.
From Euler formula, we have for real x that:

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x,$$

which gives

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This prompts the following definitions. For $z \in \mathbb{C}$, we define:

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2}} \quad \text{and} \quad \boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

we defined the other trigonometric functions the usual way

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \quad \text{and} \quad \cot(z) = \frac{\cos(z)}{\sin(z)}$$
$$\sec(z) = \frac{1}{\cos(z)} \quad \text{and} \quad \csc(z) = \frac{1}{\sin(z)}$$

Properties: For $z, w \in \mathbb{C}$ and $x, y \in \mathbb{R}$ we have:

- $\cos^2 z + \sin^2 z \equiv 1$
- $e^z = \cos z + i \sin z$
- $\cos(z + 2\pi) = \cos z$
- $\sin(z + 2\pi) = \sin z$
- $\cos(-z) = \cos z$
- $\sin(-z) = -\sin z$
- $\cos(iy) = \cosh(y)$
- $\sin(iy) = i\cosh(y)$
- $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$
- $|\cos(z)|^2 = (\cosh y)^2 - (\sin x)^2$
- $\cos(z + w) = \cos z \cos w - \sin z \sin w$
- $\sin(z + w) = \sin z \cos w + \cos z \sin w$

where $\cosh y = (e^y + e^{-y})/2$ and $\sinh y = (e^y - e^{-y})/2$ are the usual real valued hyperbolic functions.

Many of the properties familiar in the case of real trigonometric functions also hold for the complex trigonometric functions. This is not a coincidence but due to the concept of **analytic continuation** and the **identity theorems**.

Hyperbolic Functions

The real valued hyperbolic functions $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$ can also extended to the complex plane in the obvious way, by

$$\boxed{\cosh(z) = \frac{e^z + e^{-z}}{2}} \quad \text{and} \quad \boxed{\sinh(z) = \frac{e^z - e^{-z}}{2}}$$

$$\boxed{\tanh(z) = \frac{\sinh z}{\cosh z}} \quad \text{and} \quad \boxed{\coth(z) = \frac{\cosh z}{\sinh z}}$$

Properties: For $z, w \in \mathbb{C}$ and $x, y \in \mathbb{R}$ we have:

- $\cosh(z + 2\pi i) = \cosh z$
- $\sinh(z + 2\pi i) = \sinh z$
- $\cos(iz) = \cosh(z)$
- $\sin(iz) = i \sinh(z)$
- $\cosh(iz) = \cos(z)$
- $\sinh(iz) = i \sin(z)$
- $|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$
- $|\cos(x + iy)|^2 = \cos^2 x + \sinh^2 y$
- $\cosh(z + w) = \cosh z \cosh w + \sinh z \sinh w$
- $\sinh(z + w) = \sinh z \cosh w + \cosh z \sinh w$

Inverse Trigonometric & Hyperbolic Functions

The arcsine function is the solution to the equation:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Letting $v \equiv e^{iw}$, we solve the equation

$$v + \frac{1}{v} = 2iz.$$

After simple calculations we get a quadratic equation for v,

$$v^2 - 2izv - 1 = 0,$$

whose solution is given by:

$$e^{iw} = v = iz + (1 - z^2)^{1/2} = iz \pm \sqrt{1 - z^2}.$$

Keep in mind that since z is a complex variable, $(1 - z^2)^{1/2}$ is the complex square-root function which is two-valued. Solving for $w = \arcsin(z)$ in the above equation we get the multivalued function:

$$\boxed{\arcsin(z) = -i \log\left[iz + (1 - z^2)^{1/2}\right] = -i \log\left[iz \pm \sqrt{1 - z^2}\right].}$$

This identity is to be understood as a set identity, in the sense that w satisfies $\sin w = z$ if and only if w is one of the values of $-i\log\left[iz + (1-z^2)^{1/2}\right]$. To obtain a genuine function, we must restrict the domain and specify the branch. One way to do this is to draw two branch cuts, from $-\infty$ to -1 and from $+1$ to $+\infty$ along the real axis, and to specify the branch of $\sqrt{1-z^2}$ that is positive on the interval $(-1,1)$. With this branch of $\sqrt{1-z^2}$, we obtain a continuous branch $-i\,\mathrm{Log}(iz+\sqrt{1-z^2})$ of $\arcsin z$. This defines the principal value of $\arcsin$ as

$$\boxed{\mathrm{Arcsin}(z) = -i\,\mathrm{Log}\left(iz+\sqrt{1-z^2}\right)}$$

where we use $\mathrm{Log}(z)$ for the principal value of $\log(z)$ and $\sqrt{z}$ to denote the positive principal single-valued function of $z^{1/2}$.

In a similar fashion we get the other inverse trigonometric functions and their respective principal values. We summarize these functions as follows:

- $\mathrm{Arcsin}(z) = -i\,\mathrm{Log}\left(iz+\sqrt{1-z^2}\right)$
- $\mathrm{Arccos}(z) = -i\,\mathrm{Log}\left(z+i\sqrt{z^2-1}\right)$
- $\mathrm{Arctan}(z) = \dfrac{1}{2i}\,\mathrm{Log}\left(\dfrac{i-z}{i+z}\right)$
- $\mathrm{Arccot}(z) = \dfrac{1}{2i}\,\mathrm{Log}\left(\dfrac{z+i}{z-i}\right)$

Useful identities

- $\arcsin(z) + \arccos(z) = \frac{1}{2}\pi + 2\pi n, n \in \mathbb{Z}$
- $\mathrm{Arcsin}(z) + \mathrm{Arccos}(z) = \frac{1}{2}\pi$
- $\mathrm{arccot}(z) = \arctan(1/z)$
- $\mathrm{Arccot}(z) = \mathrm{Arctan}(1/z)$
- $\arctan(z) + \mathrm{arccot}(z) = \frac{1}{2}\pi + \pi n, n \in \mathbb{Z}$
- $\mathrm{Arctan}(z) + \mathrm{Arccot}(z) = \mathrm{sign}(\mathrm{Re}\, z)\frac{1}{2}\pi$

The principal branch inverse hyperbolic functions are defined by:

- $\mathrm{Arcsinh}\, z := \mathrm{Log}[z+\sqrt{z^2+1}]$
- $\mathrm{Arccosh}\, z := \mathrm{Log}[z+\sqrt{z+1}\sqrt{z-1}]$
- $\mathrm{Arctanh}\, z := \frac{1}{2}[\mathrm{Log}(1+z) - \mathrm{Log}(1-z)], z \neq \pm 1$

The multiple-valued inverse functions are obtained by replacing the Log with log and the principal branch square root functions with the two-valued square root.

3.7 Branches of multi-valued functions

In this section we will touch on the concepts of branches, branch points and branch cuts. These concepts (which are notoriously difficult to understand for beginners) are typically defined in terms functions of a complex variable. Here we will develop these ideas as they relate to $\arg z$. Our methods of investigating continuity and other properties for single-valued functions cannot be used for multiple-valued functions. Fortunately, a multiple-valued function can quite naturally be replaced by many different single-valued functions. The nature of multiple-valued function may then be examined from the point of view of its single-valued counterparts.

Definition: Let $f(z)$ be a multiple-valued complex valued function.

(1) A branch of the multiple-valued function f is any single-valued function F that is analytic in some domain D at each point z of which the value $F(z)$ is one of the values of f .

(2) A point z_0 is a **branch point** of a function $f(z)$ if the function changes value when you walk around the point on any path that encloses no singularities other than the one at $z = z_0$.

(3) The function $f(z)$ has a **branch point at infinity** if $f(1/z)$ has a branch point at 0.

(4) A **branch cut** is a curve in the complex plane such that it is possible to define a single valued analytic branch of a multi-valued function on the plane minus that curve. Branch cuts are usually, but not always, taken between pairs of branch points.

Branch points at infinity : paths around infinity. We can also check for a branch point at infinity by following a path that encloses the point at infinity and no other singularities. Just draw a simple closed curve that separates the complex plane into a bounded component that contains all the singularities of the function in the finite plane. Then, depending on orientation, the curve is a contour enclosing all the finite singularities, or the point at infinity and no other singularities.

Example. Once again consider the function $z^{1/2}$. We know that the function changes value on a curve that goes once around the origin. Such a curve can be considered to be either a path around the origin or a path around infinity. In either case the path encloses one singularity. There are branch points at the origin and at infinity. Now consider a curve that does not go around the origin. Such a curve can be considered to be either a path around neither of the branch points or both of them. In this case $z^{1/2}$ does not change value when we follow a path that encloses neither or both of its branch points.

Example. Consider $f(z) = (z^2 - 1)^{1/2}$. We factor the function.

$$f(z) = (z-1)^{1/2}(z+1)^{1/2}$$

There are branch points at $z = \pm 1$. Now consider the point at infinity.

$$f(1/z) = (z^{-2} - 1)^{1/2} = \pm z^{-1}(1 - z^2)^{1/2}$$

Since $f(1/z)$ does not have a branch point at $z = 0$, $f(z)$ does not have a branch point at infinity.

We could reach the same conclusion by considering a path around infinity. Consider a path that circles the branch points at $z = \pm 1$ once in the positive direction. Such a path circles the point at infinity once in the negative direction. In traversing this path, the value of $f(z)$ is multiplied by the factor $(e^{i2\pi})^{1/2}(e^{i2\pi})^{1/2} = (e^{i2\pi}) = 1$. Thus the value of the function does not change. There is no branch point at infinity

Branches of $\log z$

In most of the examples that we will encounter, the multi-valuedness arises ultimately from the complex logarithm $\log z$, and in such cases the branch points are the values of z such that the input to the logarithm is 0 or ∞.

Assign branch cuts in the complex plane, such that:

- Every branch point has a branch cut ending on it.
- Every branch cut ends on a branch point.

Note that any branch point lying at infinity must also obey these rules. The branch cuts should not intersect. It is worth emphasizing again that branch points are independent of the choice of branch cuts. Each branch point is, by definition, a point where the multi-valued operation becomes single-valued. For the operations z^a and $\log z$, the branch points are at $z = 0$ and $z = \infty$, but for other operations they may occur at other positions in the complex plane.

The choice of where to place branch cuts is **not unique**. Branch cuts are usually chosen to be straight lines, for simplicity, but this is not necessary. The various choices of branch cuts simply correspond to different ways of partitioning the multi-valued operation's various values into distinct branches.

Example. Consider for $z \neq 0$ the multiple-valued function

$$w = \log(z) = \ln|z| + i\arg(z) = \ln|z| + i\,\mathrm{Arg}\,z + i2k\pi,\ k \in \mathbb{Z}$$

Note that $z_0 = 0$ is **branch point** of $\log z$ because if we choose a closed path around $z_0 = 0$ the $\arg z$ will increase of decrease by $2i\pi$ at every turn.

Furthermore since $\log(1/z) = -\log(z)$ then $\log z$ has a **branch point at infinity**.

Arbitrary Branches of $\log z$. Let t is any real number. If we restrict the value of $\arg z$ in the definition $\log(z)$ so that $t < \theta = \mathrm{Arg}_t\, z < t + 2\pi$, then the function

$$\boxed{\mathrm{Log}_t(z) = \ln|z| + i\,\mathrm{Arg}_t(z);\ t < \mathrm{Arg}_t < t + 2\pi}$$

is single-valued in the stated domain and thus is a **branch** of $\log z$. Its **branch cut** is ray $\arg z = t$ and $z = 0$ is its **branch point** as seen in the figure below. Its derivative is given by

$$\frac{d}{dz}\log_t(z) = \frac{1}{z}, \quad \text{where} \quad |z| > 0,\ t < \mathrm{Arg}_t\, z < t + 2\pi.$$

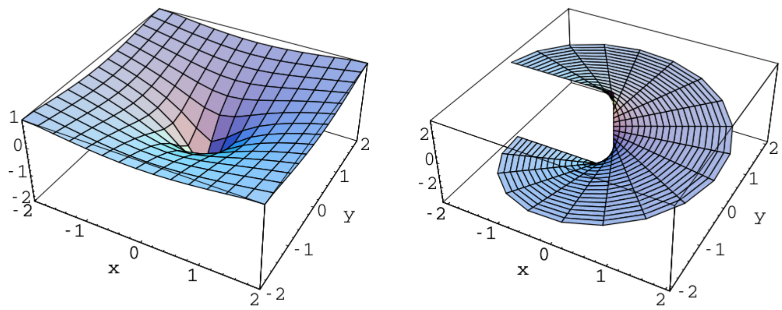
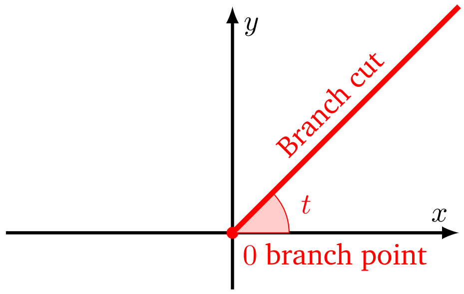


Figure 3.10: Plots of $\mathrm{Re}(\mathrm{Log}\, z) = \ln|z|$ and $\mathrm{Im}(\mathrm{Log}\, z) = \mathrm{Arg}\, z$



Remark. The *principal-value* or *principal branch* of $\log(z)$ is defined as

$$\mathrm{Log}(z) = \mathrm{Log}_{-\pi}(z) = \ln|z| + i\,\mathrm{Arg}_{-\pi}(z) = \ln|z| + i\,\mathrm{Arg}(z)$$

is a particular case of $\log_t(z)$ when $t = -\pi$.
Its domain of analyticity is $D^* = \mathbb{C} \setminus (-\infty, 0]$ (which is this is the complex plane with the negative real axis removed) where the function is single valued, analytic and its derivative is $(\mathrm{Log}(z))' = 1/z$.

Other branches of $\log z$ may be defined by restricting $\arg z$ to $(2k-1)\pi < \arg z \leq (2k+1)\pi$, k an integer. The "cut line" may not be crossed while continuously varying the argument of z without moving from one branch to another, which would destroy single-valuedness.

Example. Consider the function

$$w = f(z) = z^{1/2} = e^{\log(z)/2} = |z|^{1/2} e^{i\arg(z)/2} = |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2+ik\pi} = |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2} e^{ik\pi}.$$

If $k = 0$ we get $w_0 = |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2} = \sqrt{z}$. This is the principal branch of $w = z^{1/2}$.
If $k = 1$ we get $w_1 = |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2} e^{i\pi} = -\sqrt{z}$.
If $k = 2$ we get $w_2 = |z|^{1/2} e^{i\,\mathrm{Arg}(z)/2} e^{i2\pi} = \sqrt{z}$.

Since $z^{1/2} = e^{\log(z)/2}$ and $\log z$ has branch points at $z = 0$ and $z = \infty$, so does $w = z^{1/2}$. In general, any time we walk around the origin, the value of $z^{1/2}$ changes by the factor -1. This makes $z = 0$ a branch point of $z^{1/2}$. Thus the function changes value on a curve that goes once around the origin. Such a curve can be considered to be either a path around the origin or a path around infinity. In either case the path encloses one singularity. There are branch points at the origin and at infinity. Now consider a curve that does not go around the origin. Such a curve can be considered to be either a path around neither of the branch points or both of them. It can be show that $z^{1/2}$ does not change value when we follow a path that encloses neither or both of its branch points.

Diagnosing branch points. We have the definition of a branch point, but we do not have a convenient criterion for determining if a particular function has a branch point. We have seen that $\log z$ and z^α for non-integer α have branch points at zero and infinity. The inverse trigonometric functions like the arcsine also have branch points, but they can be written in terms of the logarithm and the square root. In fact all the elementary functions with branch points can be written in terms of the functions $\log z$ and z^α. Furthermore, note that the multi-valuedness of z^α comes from the logarithm, $z^\alpha = e^{\alpha \log z}$. This gives us a way of quickly determining if and where a function may have branch points in the result below.

> If $f(z)$ be a single-valued function. Then $\log(f(z))$ and $(f(z))^\alpha$ may have branch points only where $f(z)$ is zero or singular.

Example. Are the functions below multi-valued? Do they have branch points?
(a) $w = (z^2)^{1/2}$ (b) $w = (z^{1/2})^2$ (c) $w = (z^{1/2})^3$

(a) Notice that $w = (z^2)^{1/2} = \pm\sqrt{z^2} = \pm z$. Because the $(\cdot)^{1/2}$, the function is multi-valued. The only possible branch points are at zero and infinity.
We have $[(e^{i0})^2]^{1/2} = 1$ and $[(e^{i2\pi})^2]^{1/2} = e^{i2\pi} = 1$, thus we see that the function does not change value when we walk around the origin. We can also consider this to be a path around infinity. This function is multi-valued, but has no branch points.

(b) We have $w = (z^{1/2})^2 = (\pm\sqrt{z})^2 = z$, which is single valued.

(c) For this function we have $w = (z^{1/2})^{13} = (\pm\sqrt{z})^3 = \pm(\sqrt{z})^3$. and thus is multi-valued. The only possible branch points are at zero and infinity. We have $[(e^{i0})^{1/2}]^3 = 1$ and $[(e^{i2\pi})^{1/2}]^3 = e^{i3\pi} = -1$. Since the function changes value when we walk around the origin, it has a branch point at $z = 0$. We can also show that it has a branch point at infinity.

Example Consider the function $f(z) = \log\left(\dfrac{1}{z-1}\right)$. Since $\dfrac{1}{z-1}$ is only zero at infinity and its only singularity is at $z = 1$, the only possibilities for branch points are at $z = 1$ and $z = \infty$. Since

$$\log\left(\frac{1}{z-1}\right) = -\log(z-1)$$

and and $\log z$ has branch points at zero and infinity, we see that $f(z)$ has branch points at $z = 1$ and $z = \infty$.

Example Find the branch points if any of the following functions
(a) $w = \sin(z^{1/2})$ (b) $w = (\sin z)^{1/2}$ (c) $w = z^{1/2}\sin(z^{1/2})$ (d) $w = (\sin z^2)^{1/2}$

(a) $w = \sin(z^{1/2}) = \sin(\pm\sqrt{z}) = \pm\sin(\sqrt{z})$ and since $z^{1/2}$ has branch points at $z = 0$ and $z = \infty$ so does $\sin z^{1/2}$.

(b) The function $w = (\sin z)^{1/2} = \pm\sqrt{\sin z}$ is multi-valued. The possible branch points are at $\sin z = 0$ and $\sin z = \infty$. Since $\sin z = 0$ when $z_k = k\pi$ then $(\sin z)^{1/2}$ has branch points at $z_k = k\pi;\ k \in \mathbb{Z}$.
Since the branch points at $z = k\pi$ go all the way out to infinity. It is not possible to make a path that encloses infinity and no other singularities. The point at infinity is a non-isolated singularity. A point can be a branch point only if it is an isolated singularity.

(c) The function $w = z^{1/2}\sin(z^{1/2}) = \pm\sqrt{z}\sin(\pm\sqrt{z}) = \sqrt{z}\sin(\sqrt{z})$ is single valued. Thus there could be no branch points

(d) The function $w = (\sin z^2)^{1/2} = \pm\sqrt{\sin z^2}$ is multi-valued, its possible branch points are when $\sin z^2 = 0$ which are at $z_k = \sqrt{k\pi}$.
First we consider the case when $z = 0$. We have seen that $(z^2)^{1/2}$ does not have a branch point at $z = 0$ and thus $(\sin z^2)^{1/2}$ does not either.
Now we consider $z_k = k\pi$ with $k \in \mathbb{N}$. Since $(z - \sqrt{k\pi})^{1/2}$ has branch points at $z = \sqrt{k\pi}$ so does $(\sin z^2)^{1/2}$. Thus $w = (\sin z^2)^{1/2}$ has branch points at $z_k = \sqrt{k\pi}; k \in \mathbb{Z}^*$.
This is the set of numbers $\{\pm\sqrt{\pi}, \pm\sqrt{2\pi}, \ldots, \pm i\sqrt{\pi}, \pm i\sqrt{2\pi}, \ldots\}$.
The point at infinity is a non-isolated singularity.

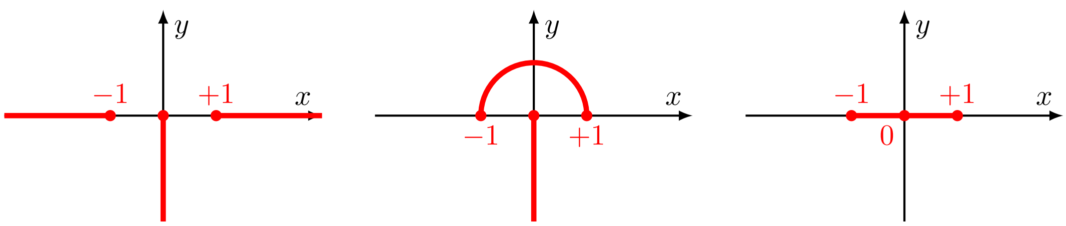
Example. Find the branch points of $w = f(z) = (z^3 - z)^{1/3}$.
If we expand $f(z)$ we get

$$w = f(z) = z^{1/3}(z-1)^{1/3}(z+1)^{1/3}$$

which has branch points at $z = -1, 0, 1$. We consider also the points at infinity

$$f(1/z) = (z^{-3} - z^{-1})^{1/3} = z^{-1}(1-z)^{1/3}(1+z)^{1/3}$$

for which $z = 0$ is not a zero and hence $z = \infty$ is not a branch point of $f(z)$.
Below some possible branch cuts of the function.



Three Possible Branch Cuts for $f(z) = (z^3 - z)^{1/3}$.

3.8 Riemann Surfaces

Consider the mapping $w = \log(z)$. Each nonzero point in the $z-$plane is mapped to an infinite number of points in the $w-$plane.

$$w = \{\ln|z| + i\arg(z)\} = \{\ln|z| + i(\mathrm{Arg}(z) + 2k\pi)|k \in \mathbb{Z}\}$$

This multi-valuedness makes it hard to work with the logarithm. We would like to select one of the branches of the logarithm. One way of doing this is to decompose the $z-$plane into an infinite number of sheets. The sheets lie above one another and are labeled with the integers, $k \in \mathbb{Z}$. (See Figure 3.11.)

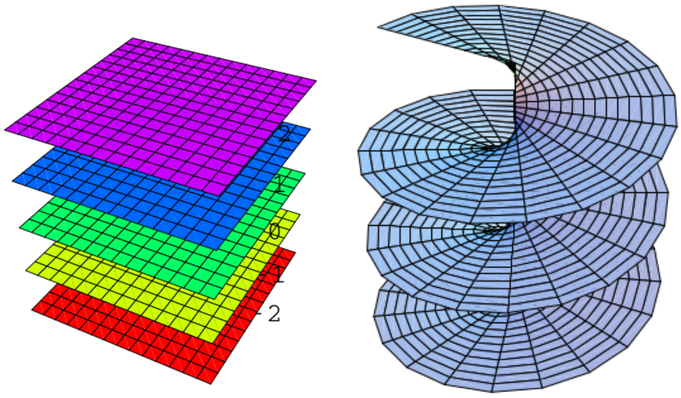


Figure 3.11: The $z-$plane decomposed into flat or corkscrew sheets

We label the point z on the $n-$th sheet as (z, n). Now each point (z, n) maps to a single point in the w-plane. For instance, we can make the zeroth sheet map to the principal branch of the logarithm. This would give us the following mapping.

$$\log(z, n) = \mathrm{Log}\, z + i2\pi k$$

This is a nice idea, but it has some problems. The mappings are not continuous. Consider the mapping on the zeroth sheet. As we approach the negative real axis from above z is mapped to $\ln|z| + i\pi$ as we approach from below it is mapped to $\ln|z| + i\pi$. The mapping is not continuous across the negative real axis. Let's go back to the regular $z-$plane for a moment. We start at the point $z = 1$ and selecting the branch of the logarithm that maps to zero $(\log(1) = i2k\pi)$. We make the logarithm vary continuously as we walk around the origin once in the positive direction and return to the point $z = 1$. Since the argument of z has increased by 2π, the value of the logarithm has changed to $2i\pi$. If we walk around the origin again we will have $\log(1) = 4i\pi$. Thus $\log(z)$ has a **branch point** at $z = 0$.

Furthermore since $\log(1/t) = -\log(t)$, we see that $\log(t)$ has a branch at $t = 0$ which implies that $\log z$ has a branch point at infinity.
Our flat sheet decomposition of the $z-$plane does not reflect this property. We need a de-

composition with a geometry that makes the mapping continuous and connects the various branches of the logarithm. Drawing inspiration from the plot of $\arg(z)$, we decompose the $z-$plane into an infinite corkscrew with axis at the origin. (See Figure 3.11.) We define the mapping so that the logarithm varies continuously on this surface. Consider a point z on one of the sheets. The value of the logarithm at that same point on the sheet directly above it is $2i\pi$ more than the original value. We call this surface, the **Riemann surface** for the logarithm. The mapping from the Riemann surface to the $w-$plane is continuous and one-to-one.

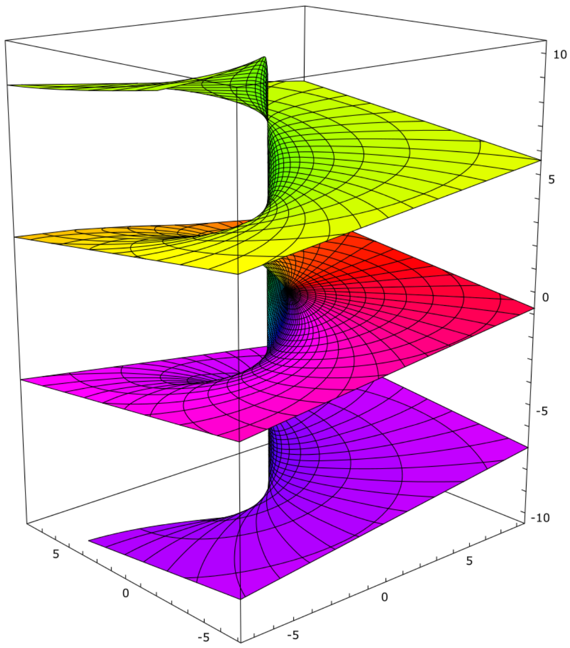
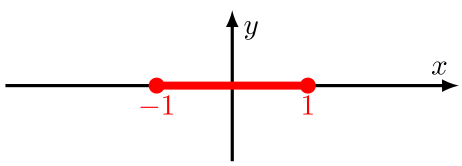


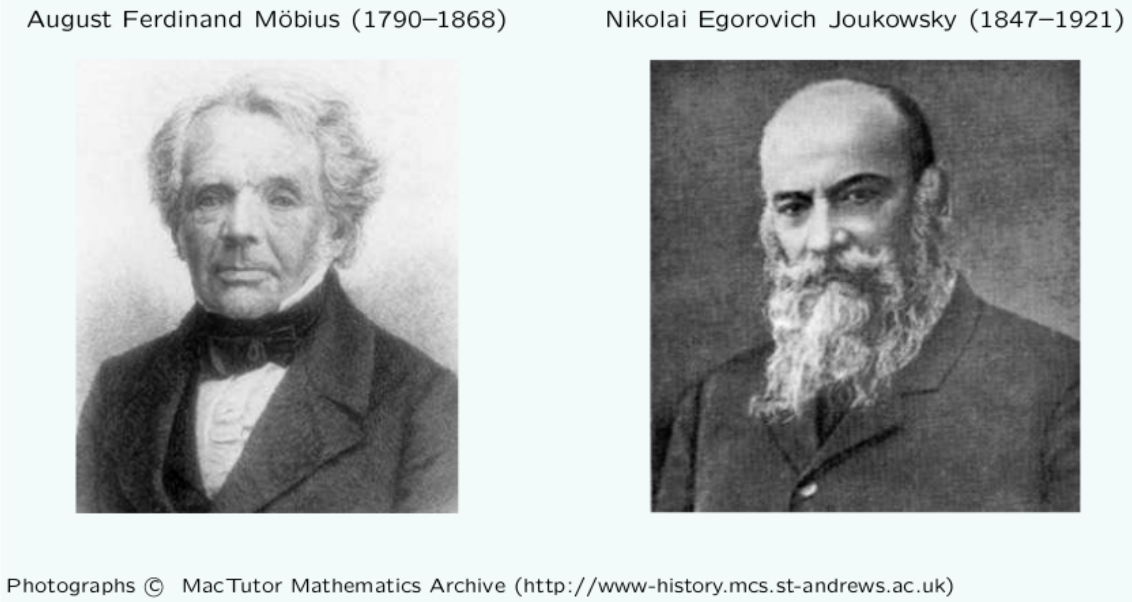
Figure 3.12: Riemann surface of $\mathrm{Im}(\log z) = \arg z$

A plot of the multi-valued imaginary part of the complex logarithm function, which shows the branches. As a complex number z goes around the origin, the imaginary part of the logarithm goes up or down. This makes the origin a branch point of the function.
The real part of the logarithm is the single-valued $\ln|z|$; the imaginary part is the multi-valued $\arg z$.

Example. Determine the domain of analyticity of $w = \mathrm{Log}(z^2 - 1)$.
Since $z^2 - 1 = (x^2 - y^2 - 1) + i(2xy)$ then w is analytic in $\mathbb{C}$ except at points where $x^2 - y^2 - 1 < 0$ and $2xy = 0$. If $x = 0$ then $\mathrm{Re}\, w = -y^2 - 1 < 0$ for all $y \in \mathbb{R}$ and if $y = 0$ then $\mathrm{Re}\, w = x^2 - 1 < 0$ for all $x \in (-1, 1)$. Hence the domain of analyticity is $\mathbb{C} \setminus [-1, 1]$



4 Conformal Maps and Bilinear Transformations



Mathematics is an independent world created out of pure intelligence.

– William Wordsworth

4.1 Analytic functions as conformal mappings

Definition: A conformal map $f : U \to V$ is a function which preserves angles (in magnitude as well as in orientation). More specifically, f is conformal at a point if the angle between any two C^1 curves through the point is preserved under the mapping.

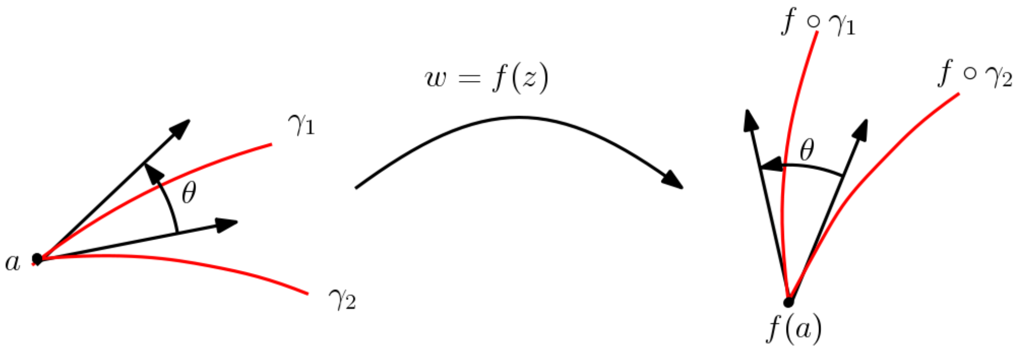


Figure 4.1: Angle preserving mappings

The following result shows where a mapping by an analytic function is conformal.

Theorem. Let f be an analytic function in the domain D, and let a be a point in D. If $f'(a) \neq 0$, then f is conformal at a.

Proof: If $\gamma : [0,1] \to \mathbb{C}$ is a C^1 curve and $f[\gamma(t)]$ its image. The tangent slopes are

$$\arg(\gamma'(t)), \gamma'(t) \neq 0; \arg[f(\gamma(t))]', [f(\gamma(t))]' = f'[\gamma(t)]\gamma'(t) \neq 0 \text{ if } f'(z) \neq 0 \text{ and } \gamma'(t) \neq 0.$$

Let $\gamma_1 : [0,1] \to \mathbb{C}$ and $\gamma_2 : [0,1] \to \mathbb{C}$ be C^1 curves through the point $z = a$ with;

$$\gamma_1(t_1) = \gamma_2(t_2) = a$$

The tangents to the curves at $z = a$ are $\arg \gamma_1'(t_1)$ and $\arg \gamma_2'(t_2)$ and the angle between them is $\arg \gamma_1'(t_1) - \arg \gamma_2'(t_2), \gamma_1'(t_1) \neq 0, \gamma_2'(t_2) \neq 0$.
Assuming $f'(a) \neq 0$ and applying the chain rule gives

$$\frac{[f(\gamma_2(t_2))]'}{[f(\gamma_1(t_1))]'} = \frac{f'(\gamma_2(t_2))\gamma_2'(t_2)}{f'(\gamma_1(t_1))\gamma_1'(t_1)} = \frac{f'(a)\gamma_2'(t_2)}{f'(a)\gamma_1'(t_1)} = \frac{\gamma_2'(t_2)}{\gamma_1'(t_1)}.$$

The result follows by taking the argument since $\arg(z_2/z_1) = \arg(z_2) - \arg(z_1)$.

Remarks.

- The above Theorem says that an analytic function is conformal at all points where the derivative is nonzero.
- If f is analytic in an open neighborhood of $z = a$ with $f'(a) \neq 0$ then by continuity, $f'(z) \neq 0$ in an open neighborhood of a, it follows that f is conformal in a neighborhood of a.
- Suppose f is conformal in a neighborhood of a and b is near a, then we have:

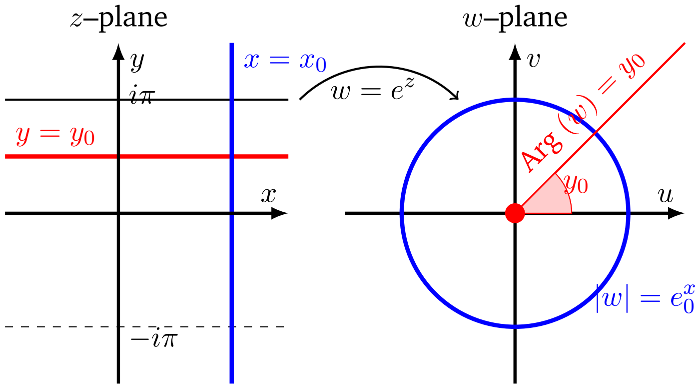
  $$w_b - w_a = f(b) - f(a) \approx f'(a)(b - a).$$

  Hence we have

  $$|w_b - w_a| \approx |f'(a)||b - a|.$$

  Therefore we say that short distances in the z plane in the neighborhood of $z = a$ are magnified (or reduced) in the w plane by an amount given approximately by $|f'(a)|$, called the linear magnification factor. Large figures in the z plane usually map into figures in the w plane that are far from similar.
- Similarly conformal mappings, transform small figures in the neighborhood of a point $z = a$ in the z plane into similar small figures in the w plane and are magnified (or reduced) by an amount given approximately by $|f'(a)|^2$, called the area magnification factor.
- Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size or curvature.

We have already discussed a number of examples of conformal maps without referring to the name "conformal". For instance, $f(z) = e^z$ is conformal on $\mathbb{C}$ and maps vertical and horizontal lines into circles and orthogonal radial rays, respectively.



Example. Show that the mapping $w = f(z) = \cos z$ is conformal at the points $a = i, 1, \pi + i$, and determine the angle of rotation given by $\arg f'(a)$ at the given points.

Solution. Since $f'(z) = -\sin z$, we conclude that the mapping $w = \cos z$ is conformal at all points except $z = n\pi$, where $n \in \mathbb{Z}$. Calculation reveals that:
$f'(i) = -i\sinh(1), f'(1) = -\sin(1)$ and $f'(\pi + i) = i\sinh(1)$.
Therefore the angle of rotation is given respectively by:
$\arg f'(i) = -\pi/2, \arg f'(1) = \pi$ and $\arg f'(\pi + i) = \pi/2$.

Example. The mapping $w = f(z) = z^2$ maps the square

$$S = \{x + iy : 0 < x < 1, 0 < y < 1\}$$

onto the region in the upper half plane $\text{Im}(w) > 0$, which lies under the parabolas $u = 1 - v^2/4$ and $u = -1 + v^2/4$, as shown in Figure. Since the derivative is $f'(z) = 2z$, and we conclude that the mapping $w = z^2$ is conformal for all $z \neq 0$. It is worthwhile to observe that the right angles at the vertices $z = 1, 1 + i, i$ are mapped onto right angles at the vertices $w = 1, 2i, -1$, respectively. At the point $z = 0$ we have $f'(0) = 0$ and $f''(0) \neq 0$. Hence angles at the vertex $z = 0$ are magnified by the factor $k = 2$. In particular, we see that the right angle at $z = 0$ is mapped onto the straight angle at $w = 0$.

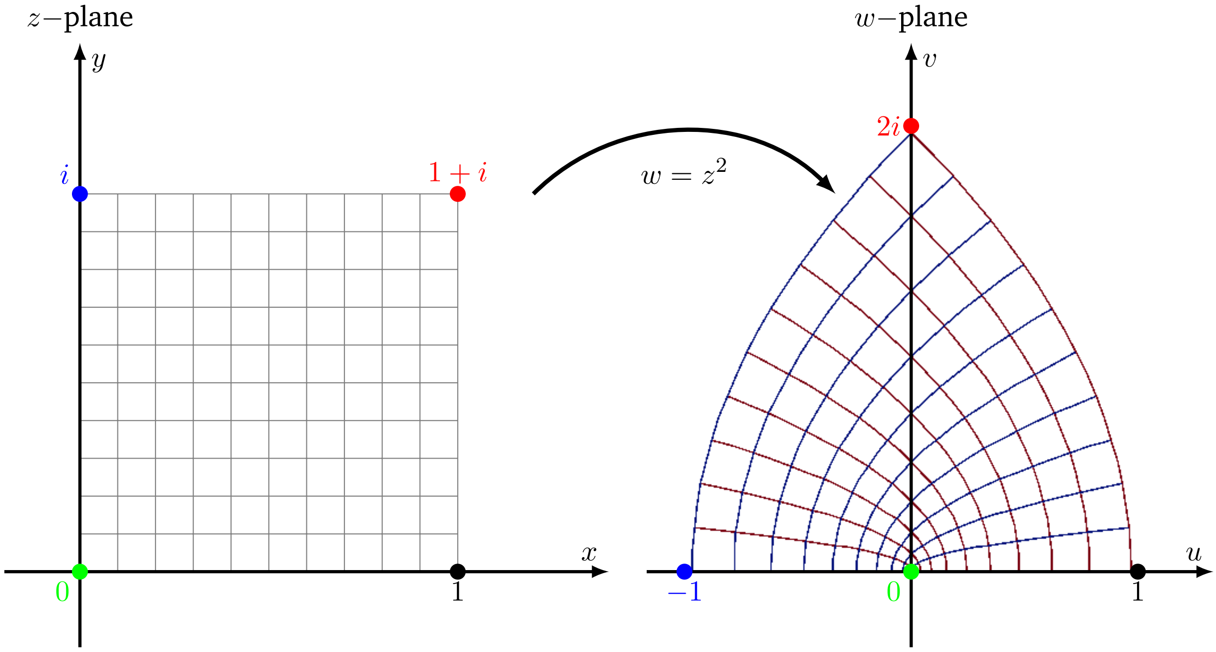


Image showing angle preserving property at points where f is conformal

Example. The function $w = f(z) = z^2$ is conformal in $\mathbb{C}^*$ because $f'(z) = 2z = 0$ only when $z = 0$. For any fixed $\theta_0, 0 < \theta < \pi/2$, f maps the sector $\{|\arg z| < \theta_0\}$ conformally onto the sector $\{|\arg z| < 2\theta_0\}$ of twice the aperture.
Hence f maps the right half-plane $\{\operatorname{Re} z > 0\}$ conformally onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$.

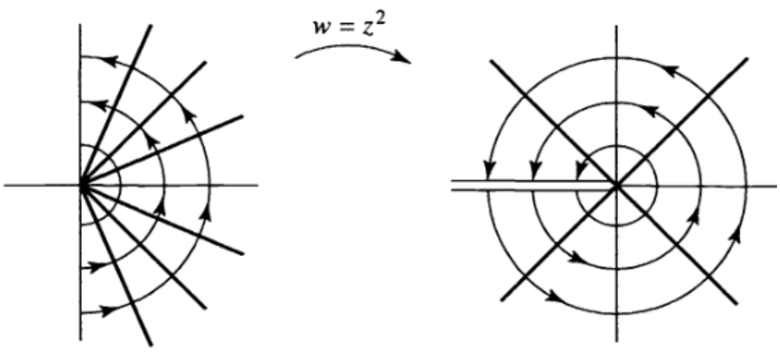


Image of $\operatorname{Im} z > 0$ under under $w = z^2$

Example. The principal branch of the logarithm $w = f(z) = \operatorname{Log} z$ is a conformal mapping of the slit plane $\mathbb{C} \setminus (-\infty, 0]$ onto the horizontal strip $\{-\pi < \operatorname{Im} w < \pi\}$.

Terminology.

(1) In complex analysis, an analytic (holomorphic) function on an open subset of the complex plane is called **univalent** if it is injective (one-to-one). Univalent functions are conformal.

(2) If U and V are two open subsets of the complex plane, then $f : U \to V$ is **bianalytic** (or biholomorphic) if it is analytic (holomorphic) and bijective. (So, implicitly, it's inverse is holomorphic.)
(3) If U is an open proper subset of $\mathbb{C}$, $U \neq \mathbb{C}$, and if U is homeomorphic to $\mathbb{D}$, then U is **conformally equivalent** to $\mathbb{D}$. That is, there is a holomorphic mapping $f : U \to \mathbb{D}$ which is one-to-one and onto.

Some properties of conformal maps

We list below some of main results about conformal mappings:

(1) If $f(z)$ is analytic and $f'(z) \neq 0$ in a region D, then the mapping $w = f(z)$ is conformal at all points of D.
(2) If $f(z)$ is analytic at $z = a$ with $f'(a) \neq 0$, then $f(z)$ is one-to-one in some neighborhood of $z = a$.
(3) If $f(z)$ is analytic and one-to-one in a domain D, then $f'(z) \neq 0$ in D, so that f is conformal on D.
(4) Let $f(z)$ be analytic in a domain D and $z = a \in D$. Then f is bi-analytic at $z = a$ iff $f'(a) \neq 0$.
(5) Let $f(z)$ be analytic in a simply connected domain D and on its boundary, the simple closed contour C. If $f(z)$ is one-to-one on C ,then $f(z)$ is one-to-one in D.
(6) Suppose $f(z)$ is analytic at $z = a$, and that the derivative $f'(z)$ has a zero of order $k - 1$ at $z = a$. If two smooth curves in the domain of f intersect at an angle θ, then their images intersect at an angle $k\theta$.
(7) (Boundary Behavior) Suppose that f is analytic and one-to-one on a region D . Then f maps the boundary of D onto the boundary of $f[D]$.

4.2 The Riemann Mapping Theorem

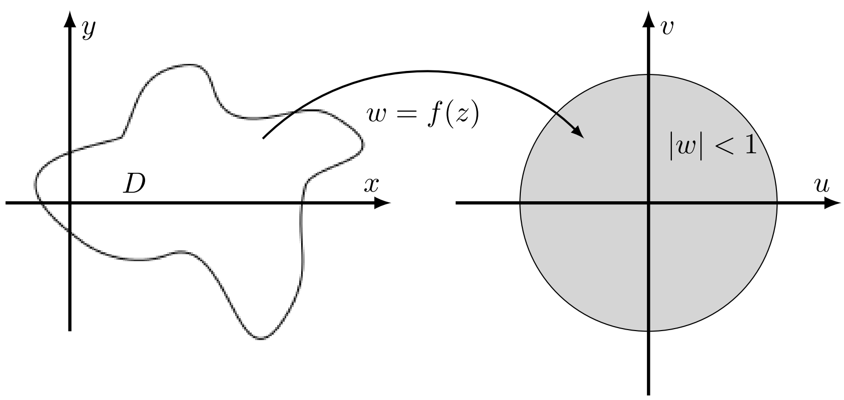
We have already discussed a number of examples of analytic functions between various domains of the complex plane. In some cases, we have given complete characterizations for mappings between certain domains such as discs and half-planes. Also, we know from the open mapping theorem that non constant analytic functions map domains into domains. Now, suppose D_1 and D_2 are simply connected domains. Then there is always almost an analytic function mapping D_1 onto D_2.
We first discuss a "typical" exception. Suppose $D_1 = \mathbb{C}$ and $D_2 = \mathbb{D}$. There can be no function analytic in the plane (entire) that maps onto the (bounded) disc $\mathbb{D}$, for, according to Liouville's theorem, constant functions are the only entire functions whose images are contained in the disc. Our major theorem of this section says that a one-to-one analytic mapping exists between any two simply connected domains, neither of which is the whole plane.

The Riemann Mapping Theorem

Let D be a nonempty proper ($D \subsetneq \mathbb{C}$) simply connected open subset of $\mathbb{C}$, and let $c \in D$. Then there exists a unique one-to-one analytic function $f : D \to \mathbb{D}$ such that $f(c) = 0, f'(c) > 0$ and $f(D) = \mathbb{D}$.

The proof of the Riemann mapping theorem extends beyond the scope of these notes. A proof of this theorem can be found in L. Ahlfors: Complex Analysis, 3rd Ed., Inter. Ser. in Pure & Applied Math. McGraw-Hill Ed, 1979 or in J. Conway: Functions of One Complex Variable I, GTM 11, 2nd Ed., Springer 1978.

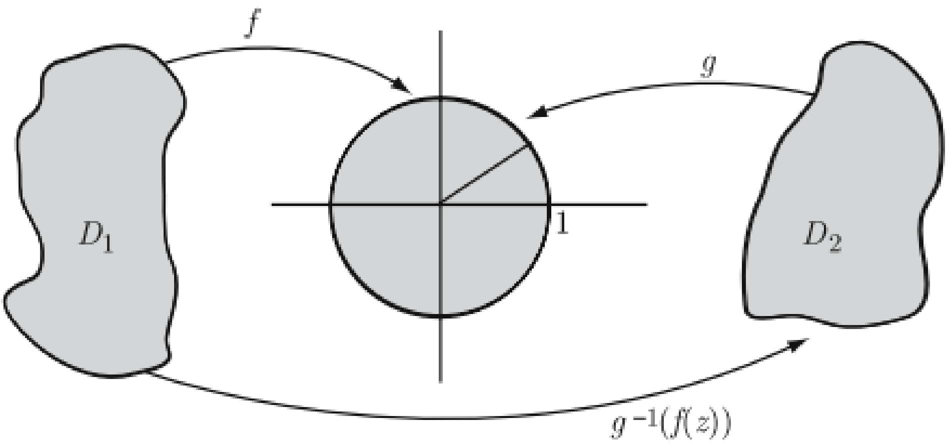
The RMP states that any proper open simply connected subset of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$.



Riemann Mapping Theorem

Corollary. If D is a nonempty simply connected domain in $\mathbb{C}_\infty$, then D is conformally equivalent to one and only one of the following domains:
(i) $\mathbb{C}_\infty$ if the boundary of D consists of no points.
(ii) $\mathbb{C}$ if the boundary of D consists of one point.
(iii) $\mathbb{D}$ if the boundary of D consists of more than one point.

Remarks.
(1) A non-constant analytic function maps open connected sets to open connected sets.
(2) Since a one-to-one analytic map is invertible, it follows that any open simply-connected domain can be mapped onto any other open simply-connected domain (by a passage through $\mathbb{D}$) provided neither is $\mathbb{C}$.
(3) The Riemann Mapping Theorem (RMP) does not give a practical algorithm for finding the actual mapping.



Riemann Mapping Theorem between two simply connected sets

4.3 The Linear and Inversion Mappings

In this section we study special case of bilinear mappings, namely:
- Translations
- Rotations
- Dilations (Scaling)
- Inversions

These transformations will play a major role in explaining the behavior of Möbius transformations.

The Function $w = az + b$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ such that $z \mapsto w = f(z) = az + b$ where a and b are constant complex numbers. We have seen previously that if $a \neq 0$ then f is one-to-one and onto. We will study the family of these functions in separate cases.

Case 1. If $a = 0$ and $b \in \mathbb{C}$ then the function $w = f(z) = b$ is the **constant** function and we have $f(\mathbb{C}) = \{b\}$. So the whole complex $z-$plane is mapped to a single point $w = b$ in the $w-$plane. Obviously in this case the function is neither one-to-one no onto.

Case 2. If $a = 1$ and $b \in \mathbb{C}$ then we have $w = f(z) = z + b$ which maps any sets in the z-plane onto a set in the w-plane displaced through the vector b. This mapping is known as a **translation**. Note that the set in the w plane will have the same shape and size as the set in the z-plane.
For instance, the function $w = z + (1 + 2i)$ maps the square having vertices $0, 1, 1 + i$ and i onto a square having vertices $1 + 2i, 2 + 2i, 2 + 3i$, and $1 + 3i$.
To show this, let $z = x + iy$ and $w = u + iv$. Then $u + iv = (x + iy) + (1 + 2i)$, i.e. , $u = x + 1, v = y + 2$. As x describes the interval $[0, 1]$, u describes the interval $[1, 2]$; as y describes the interval $[0, 1]$, v describes the interval $[2, 3]$.

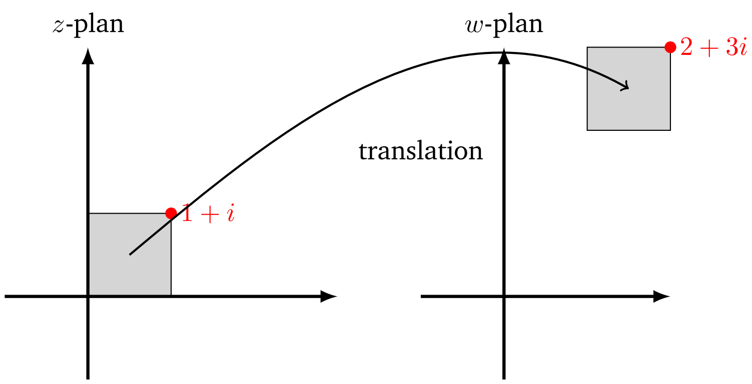


Figure 4.2: The translation $w = z + (1 + 2i)$.

Case 3. If $a > 0, a \neq 1$ and $b = 0$ then we have $w = f(z) = az = ax + iay$ is known as a **dilation** or **rescaling** and maps any set of the z-plane onto a set in the w plane scaled by a factor of a. Note that

$$|w_1 - w_2| = |f(z_1) - f(z_2)| = |a||z_1 - z_2|$$

so that the distance between any two points is multiplied by $|a|$.
If $a > 1$ we call the mapping a **magnification** and if $0 < a < 1$ we call it a **contraction**. The image in the w-plane is a scaled shape of the set in the z-plane. The image below show the mapping of the square with vertices $\{0, 1, 1 + i, i\}$ by the function $w = 2z$, which is the square with vertices $\{0, 2, 2 + 2i, 2i\}$.

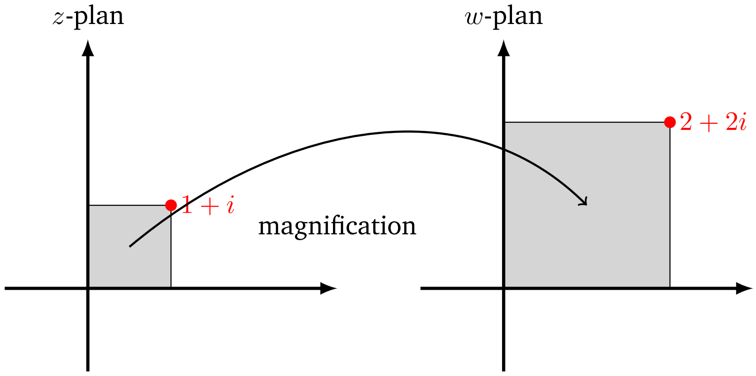


Figure 4.3: The magnification $w = 2z$.

Case 4. If $a = e^{i\alpha}, a \neq 1$ and $b = 0$ then $w = f(z) = az = e^{i\alpha}z = |z|e^{i(\arg z+\alpha)}$. Hence we have $|w| = |z|$ and $\arg(w) = \alpha + \arg(z)$, so the mapping is simply a **rotation** by the angle α. Note that $|a| = 1$, so any set of the z-plane is mapped onto a set in the w plane rotated by an angle α. The figure below shows action the mapping $w = e^{i\pi/4z}z$ on the square with vertices $\{0, 1, 1+i, i\}$.

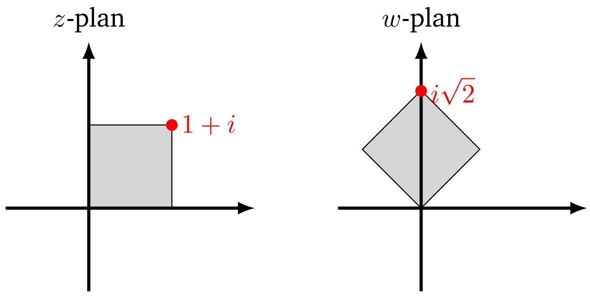


Figure 4.4: The rotation $w = e^{i\pi/4}z$.

Case 5. If $a = |a|e^{i\,\mathrm{Arg}\,a} \neq 0$ and $b \in \mathbb{C}$ then

$$f(z) = az + b = |a|e^{i\,\mathrm{Arg}\,a}z + b = (f_3 \circ f_2 \circ f_1)(z)$$

where $f_1(z) = e^{i\,\mathrm{Arg}\,a}z$ is a rotation, $f_2(z) = |a|z$ is dilation and $f_3(z) = z + b$ is a translation.Furthermore, it is one-to-one and onto.

Example 4.1. Find the image square with vertices $\{0, 1, 1+i, i\}$ under the linear mapping $w = f(z) = 2e^{i\pi/4}z - 2i$.

Solution. We have

$$f(z) = 2e^{i\pi/4}z - 2i = (f_3 \circ f_2 \circ f_1)(z)$$

where $f_1(z) = e^{i\pi/4}z$ is a rotation by and angle of $\pi/4$, $f_2(z) = 2z$ is dilation by a factor of 2 and $f_3(z) = z - 2i$ is a translation by $-2i$.

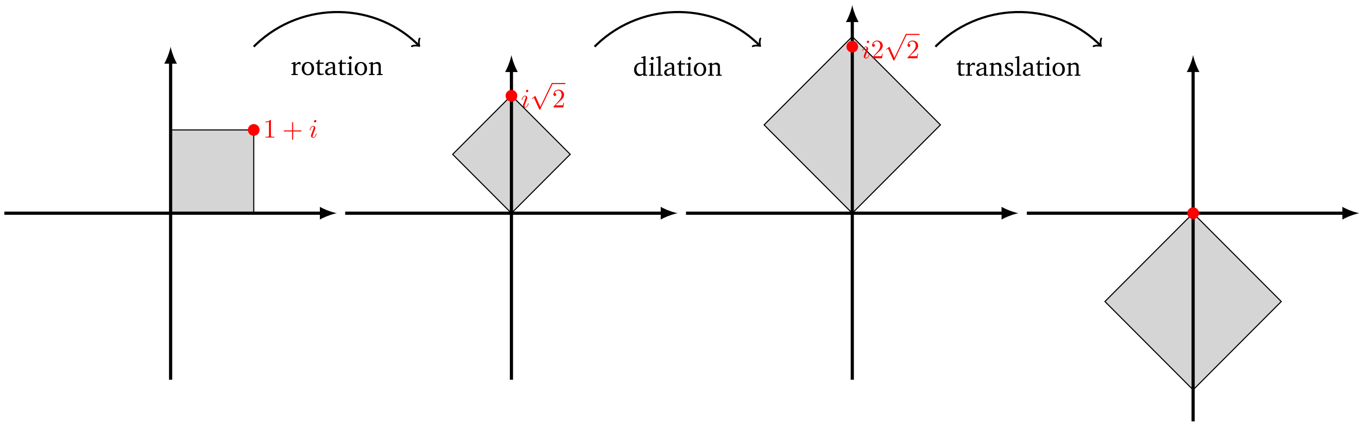


Figure 4.5: Linear Transformation $w = f(z) = 2e^{i\pi/4}z - 2i$.

There is a relationship between a complex linear function and the more familiar real-valued linear function $y = ax+b$, a straight line. The complex- valued function $w = az+b$, with a and b are complex constants, maps straight lines in of $z-$plane onto straight lines in the w-plane. Note that the complex linear functions $(a \neq 0)$ always map ∞ to ∞. We leave the determination of the effect of the constants a and b on the slope of the image line as an exercise for the reader. Observe that $w = az+b$, like its real-valued counterpart, is a one-to-one function.

Properties of linear maps

If $a \neq 0$ then the linear function $w = f(z) = az + b$ maps:
(1) Lines of the $z-$plane onto lines of the $w-$plane.
(2) Circles of the $z-$plane onto circles of the $w-$plane.
(3) Regions of the $z-$plane to geometrically similar regions of the $w-$plane.

The Function $w = 1/z$

Let us now consider the inversion function $w = f(z) = 1/z$.
This function can be considered a function of the type $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, where $\mathbb{C}_\infty$ denotes the extended complex plane. We write formally $f(0) = \infty$ and $f(\infty) = 0$.
The function $w = f(z) = 1/z$ maps points close to the origin in the $z-$plane onto points far from the origin in the $w-$plane and points far from the origin in the z plane onto points close to the origin in the w plane. In particular, as z approaches the origin, w approaches the point at ∞ in the extended complex plane. We thus have a one-to-one map from the extended plane $\mathbb{C}_\infty$ onto itself with the origin being mapped onto the point at ∞.
If we let $z = re^{i\theta}$,then $w = (1/r)e^{-i\theta}$. Thus we have $|w| = 1/r = 1/|z|$ and $\arg w = -\theta = -\arg z$. Clearly circles of the $z-$plane centered at the origin and radius r are mapped by the inversion map onto circles of the $w-$plane centered at the origin and radius $1/r$. There is also a certain symmetry with respect to both the unit circle and the real axis.

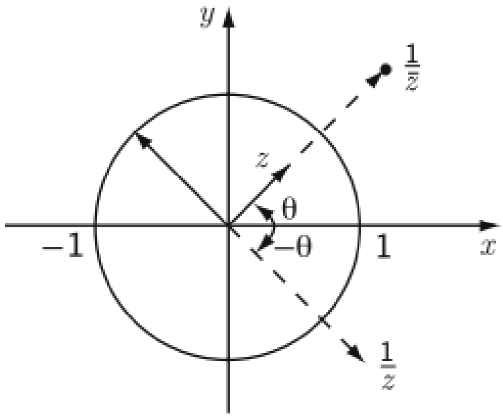


Figure 4.6: Inversion map $w = 1/z$

Points inside (outside) the unit circle are mapped onto points outside (inside) the unit circle, and points above (below) the real axis are mapped onto points below (above) the real axis (see Figure).

If we let $z = x + iy$ then we have

$$w = u + iv = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}; \quad u = \frac{x}{x^2 + y^2}; \quad v = \frac{-y}{x^2 + y^2}.$$

But since $z = 1/w$ we also get by symmetry

$$z = x + iy = \frac{1}{w} = \frac{\bar{w}}{|w|^2} = \frac{u - iv}{u^2 + v^2}; \quad x = \frac{u}{u^2 + v^2}; \quad y = \frac{-v}{u^2 + v^2}.$$

Moreover since $wz = 1$ then we have

$$|w|^2|z|^2 = (u^2 + v^2)(x^2 + y^2) = 1.$$

Now consider the equation

$$a(x^2 + y^2) + bx + cy + d = 0$$

where $a, b, c,$ and d are real constants. This equation represents a circle if $a \neq 0$ and a straight line if $a = 0$. If we multiply the above equation by $(u^2 + v^2)$ we get

$$a(x^2 + y^2)(u^2 + v^2) + bx(u^2 + v^2) + cy(u^2 + v^2) + d(u^2 + v^2) = 0$$

which show that the function $w = 1/z$ maps the equation onto the set

$$d(u^2 + v^2) + bu - cv + a = 0$$

which describes a circle for $d \neq 0$ and a straight line if $d = 0$.

Theorem 4.2. *The inversion function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$, given by $w = f(z) = 1/z$ for every non-zero $z \in \mathbb{C}$, and $f(0) = \infty$ and $f(\infty) = 0$, is one-to-one and onto. On the other hand, its inverse function is itself. Furthermore, the image under this function of a line or a circle in $\mathbb{C}_\infty$ is also a line or a circle in $\mathbb{C}_\infty$.*

Properties of the inversion map $w = 1/z$

(1) The origin maps onto the point at ∞.
(2) The point at ∞ maps onto the origin.
(3) Its inverse is itself.
(4) Every straight line passes through the point at ∞.
(5) No circle passes through the point at ∞.
(6) Circles not passing through the origin (that is, with $a \neq 0$ and $d \neq 0$) are mapped onto circles not passing through the origin.
(7) Circles passing through the origin (that is, with $a \neq 0$ and $d = 0$)are mapped onto straight lines not passing through the origin.
(8) Straight lines not passing through the origin (that is, with $a = 0$ and $d \neq 0$) are mapped onto circles passing through the origin.
(9) Straight line passing through the origin (that is, with $a = 0$ and $d = 0$) are mapped onto straight lines passing through the origin.
(10) The inversion $w = 1/z$ maps circles and straight lines onto circles and straight lines.
(11) The circle $|z| = 1$ maps onto the circle $|w| = 1$.
(12) The punctured disc $\mathbb{D} \setminus \{0\}$ maps onto $\mathbb{C} \setminus \overline{\mathbb{D}}$, and conversely.
(13) All points on $\mathbb{C} \setminus \overline{\mathbb{D}}$ map onto $\mathbb{D} \setminus \{0\}$.
(14) The interior of a circle containing the origin maps onto the exterior of a circle.
(15) The interior of a circle not containing the origin (nor having the origin as a boundary point) maps onto the interior of a circle.

Example. Show that the image of the right half plane Re $z > 1/2$, under the mapping $w = 1/z$, is the disc $|w - 1| < 1$.

Solution. We have seen that under the transformation $w = \dfrac{1}{z}$; $x = \dfrac{u}{u^2 + v^2}$. So

$$\text{Re}\, z > \frac{1}{2} \Rightarrow \frac{u}{u^2 + v^2} > \frac{1}{2} \Rightarrow u^2 - 2u + 1 + v^2 < 1 \Rightarrow (u-1)^2 + v^2 < 1 \Rightarrow |w - 1| < 1.$$

which is an inequality that determines the set of points in the w plane that lie inside the circle $|w - 1| = 1$. Since the reciprocal transformation is one-to-one, preimages of the points in the disc $|w - 1| < 1$ will lie in the right half plane Re $z > 1/2$.

Example. Find the images of the vertical lines $x = a$ and the horizontal lines $y = b$ under the mapping $w = 1/z$.

Solution. The image of the line $x = 0$ is the line $u = 0$; that is, the $y-$axis is mapped onto the $v-$axis. Similarly, the $x-$axis is mapped onto the $u-$axis. If $a \neq 0$, then we see that

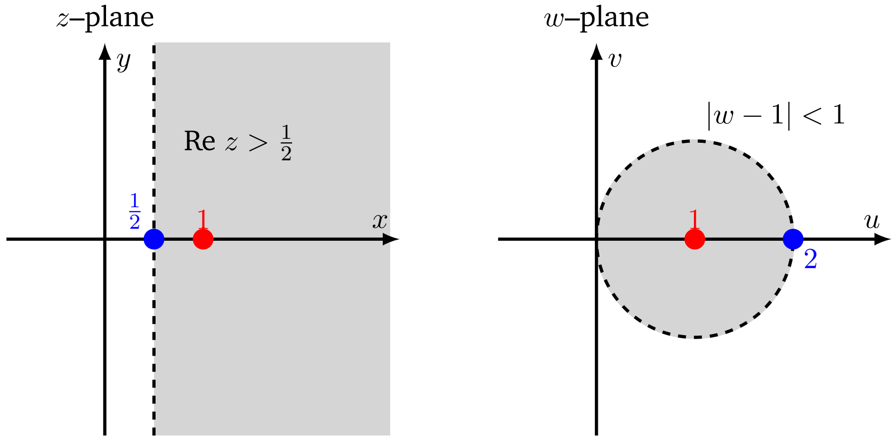


Figure 4.7: Image of Re $z > 1/2$ under the mapping $w = 1/z$.

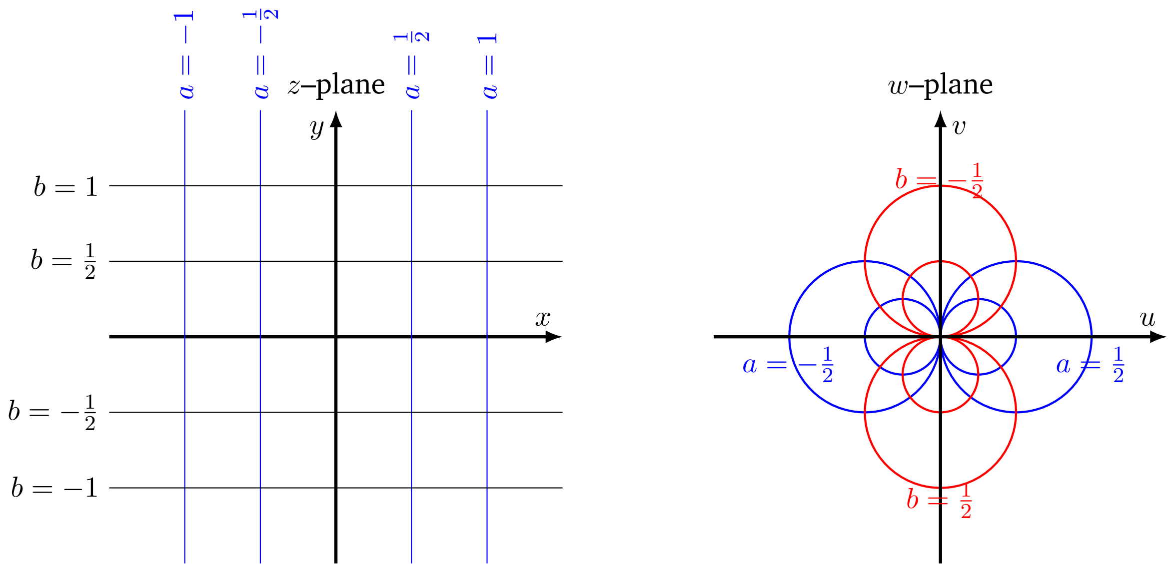


Figure 4.8: Images of horizontal and vertical lines under $w = 1/z$.

the vertical line $x = a$ is mapped onto

$$\frac{u}{u^2+v^2} = a \Leftrightarrow \left(u - \frac{1}{2a}\right)^2 + v^2 = \left(\frac{1}{2a}\right)^2$$

which is the equation of a circle in the w plane $\left|w - \frac{1}{2a}\right| = \frac{1}{2|a|}$.
Similarly, the horizontal line $y = b \neq 0$ is mapped onto the circle

$$\frac{-v}{u^2+v^2} = b \Leftrightarrow u^2 + \left(v + \frac{1}{2b}\right)^2 = \left(\frac{1}{2b}\right)^2$$

which is the equation of a circle in the $w-$plane $\left|w + \frac{i}{2b}\right| = \frac{1}{2|b|}$.

4.4 Möbius Transformations

An important class of elementary mappings was studied by Augustus Ferdinand Möbius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions and are commonly known as linear fractional or bilinear transformations. In this section we will show how they are used to map a disc one-to-one and onto a half plane.

Definition A Möbius transformation (linear fractional or bilinear transformation) is any non-constant function on $\mathbb{C}_\infty$ of the form

$$\boxed{w = T(z) = \frac{az+b}{cz+d}, \quad ad \neq bc, \quad a,b,c,d \in \mathbb{C}.}$$

Properties of Fractional Linear Transformations

Let $w = T(z) = \dfrac{az+b}{cz+d} \quad ad \neq bc, \quad a,b,c,d \in \mathbb{C}$ then we have the following:

(1) If $ad = bc$ then T would yield a constant.

(2) The coefficients are not unique, since we can multiply them all by any nonzero complex constant.

(3) To each Möbius transformation we can associate the nonsingular matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of its coefficients, which is determined up to a non-zero multiple.

(4) The linear (but non-constant) polynomials $w = Az + B, A \neq 0$ are special cases of Möbius transformations. Thus translations, rotations and dilations are special cases of Möbius transformations.

(5) The inversion mapping $w = 1/z$ is special cases of Möbius transformations.

(6) If $z \neq -d/c$ then $T(z) \in \mathbb{C}$ and $T'(z) = (ad-bc)/(cz+d)^2 \neq 0$.

(7) T is conformal and thus one-to-one and onto in $\mathbb{C} \setminus \{-d/c\}$ where $c \neq 0$.

(8) T admits an inverse, given by:

$$\boxed{T^{-1}(w) = \frac{dw-b}{-cw+a} \quad \text{if } \; w \neq a/c, w \neq \infty.}$$

(9) If $c \neq 0$ we may extend the definition of $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ to the extended complex plane as follows:

$$T(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } z \neq -d/c, z \neq \infty \\ \infty & \text{if } z = -d/c \\ \dfrac{a}{c} & \text{if } z = \infty \end{cases}$$

and T defined in this way is then one-to-one onto the extended complex plane.

(10) The inverse $T^{-1} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is

$$T^{-1}(w) = \begin{cases} \dfrac{dw-b}{-cw+a} & \text{if } w \neq a/c, w \neq \infty \\ \infty & \text{if } z = a/c \\ -\dfrac{d}{c} & \text{if } z = \infty \end{cases}$$

(11) If S and T are Möbius transformations, then so is $S \circ T$, its coefficient matrix being the product of the coefficient matrices of S and T.

(12) If T is a non-linear Möbius transformation we can rewrite it as

$$T(z) = \frac{Az+B}{z+D} = A + \frac{(B-AD)}{z+D}$$

Thus any Möbius transformation can be written as a composition of a translation, a rotation, a dilation and an inversion.

(13) Any Möbius transformation maps lines and circles onto lines and circles.

(14) Any Möbius transformation is orientation preserving in the sense that, if we traverse a circle in the order of three distinct points on it, z_1, z_2, z_3, the region to the left of the circle will map to the region to the left of the image circle, with respect to the image orientation.

(15) T preserves the property of two points being symmetric with respect to a circle, i.e., lying on the same ray from the center, and such that the geometric mean of their distances from the center equals the radius.

(16) A Möbius transform different from the identity has either one or two fixed points, as a map defined on the extended plane.

(17) A Möbius transform that leaves three distinct points invariant is the identity.

(18) Given three distinct points, z_1, z_2 and z_3 in the extended z plane and three distinct points $w_1, w2$ and w_3 in the extended w plane, there exists a unique bilinear trans-

formation $w = T(z)$ such that $T(z_1) = w_1, T(z_2) = w_2$ and $T(z_3) = w_3$. We denote that by $T\{z_1, z_2, z_3\} = \{w_1, w_2, w_3\}$. An implicit formula for the transformation is given by the equation:

$$\frac{w - w_1}{w - w_3}\frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}$$

(19) There is a unique Möbius transformation T such that $T\{z_1, z_2, z_3\} = \{0, 1, \infty\}$ and it is given by :

$$w = T(z) = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}.$$

Example 4.3. Find the FLT $T(z)$ such that $T\{0, i, \infty\} = \{-1, 0, 1\}$.

Solution. If $T(z) = \frac{az + b}{cz + d}$ then $T(\infty) = a/c = 1$ yields $a = c = 1$ and thus we can write the fractional linear transformation in the form $T(z) = w = \frac{z - i}{z + d}$.
The condition $T(0) = -1$ yields $d = i$. Thus

$$T(z) = \frac{z - i}{z + i}.$$

Example 4.4. Find the FLT $T(z)$ such that $T\{1 - i, 1 + i, -1 + i\} = \{0, 1, \infty\}$.

Solution. We can write the fractional linear transformation in the form

$$T(z) = w = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1} = \frac{z - (1 - i)}{z - (-1 + i)}\frac{(1 + i) - (-1 + i)}{(1 + i) - (1 - i)} = \frac{z - (1 - i)}{iz + (1 + i)}.$$

Example 4.5. Find the FLT $T(z)$ such that $T\{i, 2, -2\} = \{i, 1, -1\}$.

Solution. We use the equation

$$\frac{(w - i)(1 + 1)}{(w + 1)(1 - i)} = \frac{(z - i)(2 + 2)}{(z + 2)(2 - i)}$$

then solve for w to get

$$w = \frac{3z + 2i}{iz + 6}.$$

Example 4.6. Find a bilinear transformation which maps the disc $|z + i| < 1$ onto the exterior disc $|w| > 4$.

Solution. Let $T(z) = \frac{az + b}{cz + d}$ and assume that $T(-i) = \infty$, then

$$T(z) = \frac{az + b}{z + i}.$$

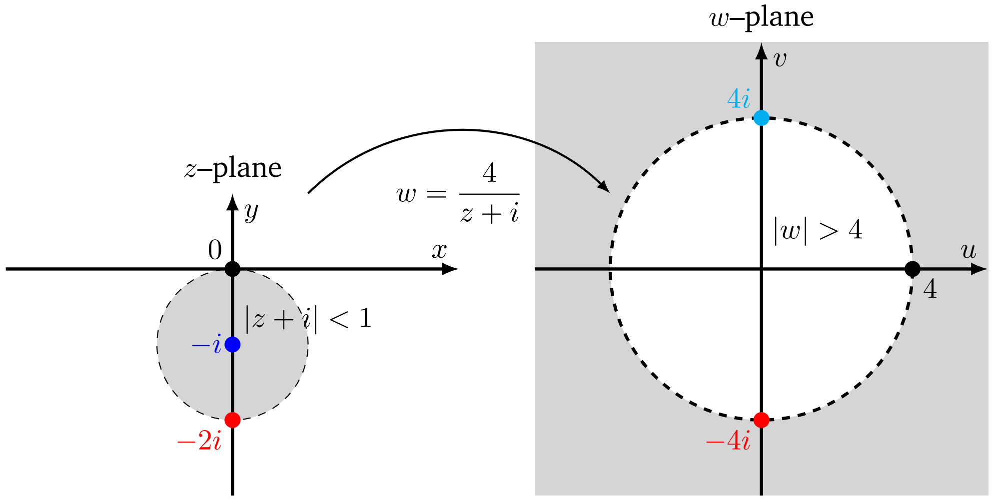


Figure 4.9: Mapping of $|z - i| < 1$ onto $|w| > 4$.

Note that $T(0) = -ib$ and $T(-2i) = 2a + ib$. These images must lie on the circle $|w| = 4$. This gives $|b| = 4$ and $|2a + ib| = \sqrt{4a^2 + b^2} = 4$. A choice satisfying the two conditions are $b = 4$ and $a = 0$. Hence

$$T(z) = \frac{4}{z+i}.$$

Example 4.7. Show that $T(z) = \dfrac{1-z}{1+z}$ maps the right half plane onto the unit disc.

Solution. Notice that $T\{i, 0, -i\} = \{-i, 1, i\}$. That is the points from the boundary of the right half plane $\{i, 0, -i\}$ get mapped onto the points $\{-i, 1, i\}$ which constitute the boundary of the unit disc. As T is one-to-one, it maps imaginary axis $\operatorname{Re} z = 0$ onto the unit circle $|w| = 1$. The image of right half plane is either the interior or the exterior of the unit circle $|w| = 1$. Now checking a point in the right half plane like $T(1/2) = 1/3 < 1$ shows that indeed the right half plane $\operatorname{Re} z > 0$ is mapped onto $|w| < 1$.

Example 4.8. Show that $T(z) = i\dfrac{1+z}{1-z}$ maps the unit disc onto the upper half plane.

Solution.
The image of the unit circle $|z| = 1$ is a line in the w plane because the point $z = 1$ belongs to the unit circle and $T(1) = \infty$. Since $T\{-1, -i, 1\} = \{0, 1, \infty\}$ then the circle $|z| = 1$ is mapped onto the real line $\operatorname{Im} w = 0$ which is the $u-$axis of the $w-$plane. So either the unit disc $|z| < 1$ is mapped onto the upper half plane $\operatorname{Im} z > 0$ or the lower half plane $\operatorname{Im} w < 0$. Since $T(0) = i$ then $|z| < 1$ is mapped onto the upper half plane $\operatorname{Im} w > 0$.

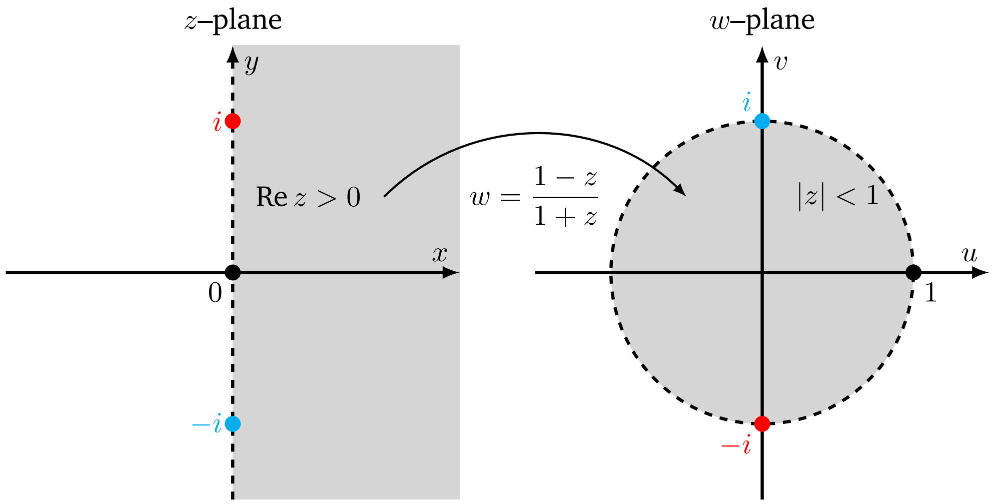


Figure 4.10: Mapping of the right half-plane onto the unit disc

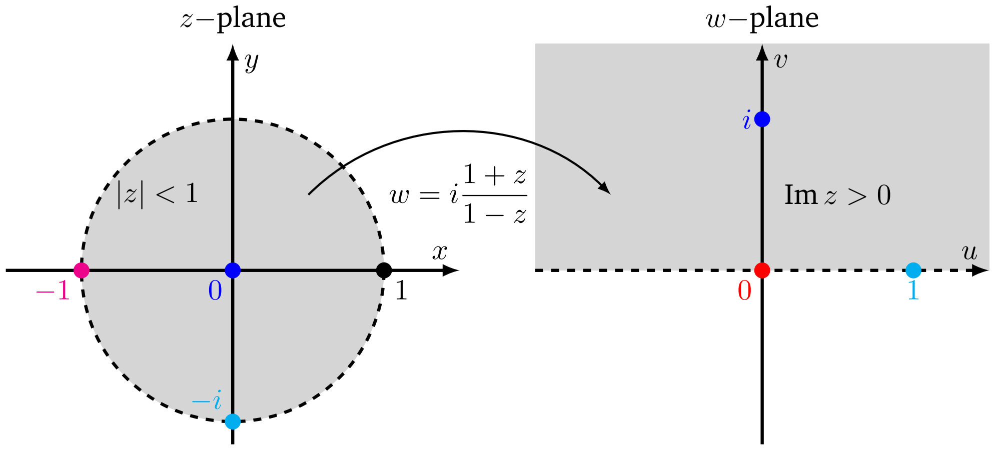


Figure 4.11: Mapping of the unit disc onto the upper half-plane.

4.5 Mapping between half planes and the unit disc

A very important class of Möbius transformation are those which map a half plane from the z plane onto the unit disc $|w| < 1$ of the w plane. The boundary of a half plane is usually a line. We have seen in chapter 1 if z_1 and z_2 are two distinct points of the complex plane $\mathbb{C}$, then the set

$$A = \{z \in \mathbb{C} : |z - z_1| = |z - z_2|\}$$

represents the set of points on the line bisecting the line segment whose end points are z_1 and z_2. Clearly if we divide by $|z - b|$ then we can re-write

$$A = \left\{ z \in \mathbb{C} : \frac{|z - z_1|}{|z - z_2|} = 1 \right\}.$$

Let $w = T(z) = (z - z_1)/(z - z_1)$ then for every $z \in A$ we have $|T(z)| = 1$. We can draw the following conclusions:

(1) T maps A onto the unit circle i.e. $T(A) = \{w \in \mathbb{C} : |w| = 1\}$.

(2) $T(z_1) = 0$ then according to the RMT the half-plane containing the point z_1 is mapped onto the unit disc $|w| < 1$.

(3) $T(z_2) = \infty$ then the half-plane containing the point z_2 is mapped onto the exterior of the closed unit disc $\{w \in \mathbb{C} : |w| > 1\}$.

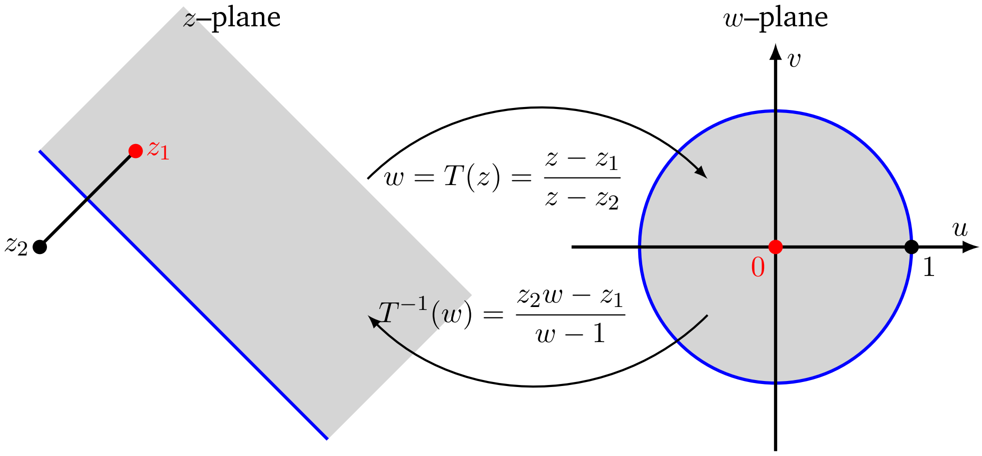


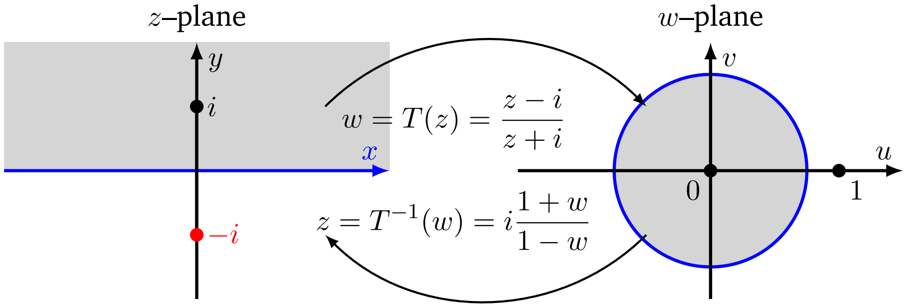
Figure 4.12: Bilinear mappings between a half-plane and the unit disc

Since T is one-to-one and onto then its inverse T^{-1} is well defined and maps the unit disc of the z plane onto the w half plane containing the point z_1.

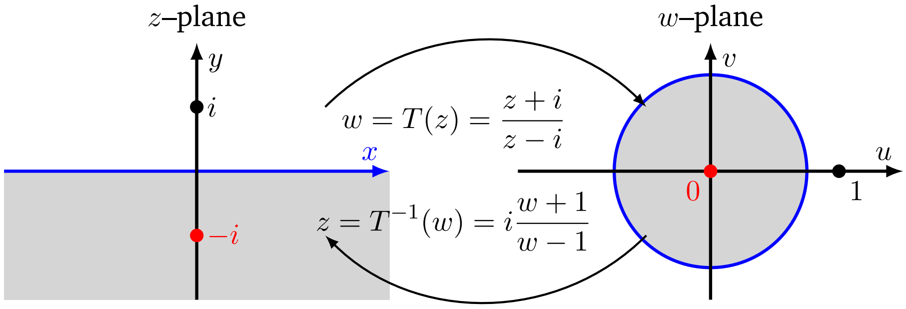
Some important bilinear mappings

We list below the main bilinear mappings between some key half planes and the unit disc.The boundary of the upper and lower half plane is the real axis $\mathbb{R}$ and the boundary of the right and left half plane is the imaginary axis $i\mathbb{R}$. The points i and $-i$ are symmetric about the the real axis and 1 and -1 are symmetric about the imaginary axis. Using these points and their respective half plane we can explicitly write the the bilinear mappings between half planes and the unit disc. Furthermore, these bilinear mappings are one-to-one and onto. Therefore if let us say $w = T(z)$ is the mapping of the upper half plane onto the unit disc, then its inverse $z = T^{-1}(w)$ would be the mapping of the unit disc onto the upper half plane.

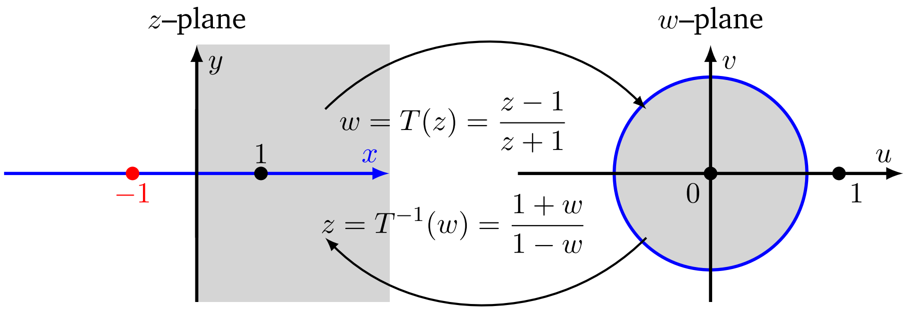
1. Bilinear mapping between the upper half plan and the unit disc.
The mapping $w = T(z) = \dfrac{z - i}{z + i}$ is known as the **Cayley** transform and maps the upper half plane onto the the unit disc. Obviously its inverse $z = T^{-1}(w) = i\dfrac{1 + w}{1 - w}$ maps the unit disc onto the upper half plane.



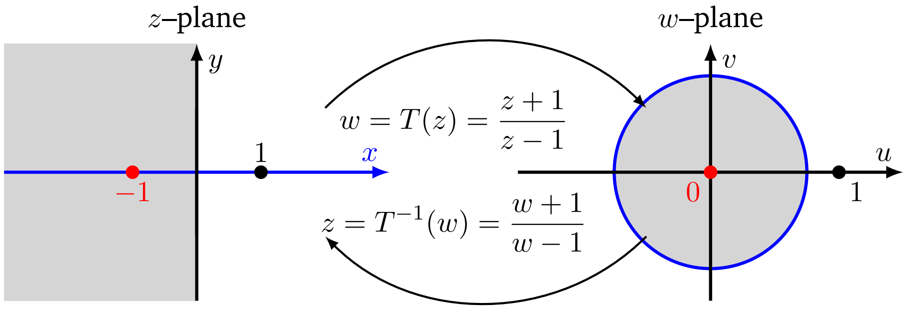
2. Bilinear mapping between the lower half plan and the unit disc.



3. Bilinear mapping between the right half plan and the unit disc.



4. Bilinear mapping between the left half plan and the unit disc.



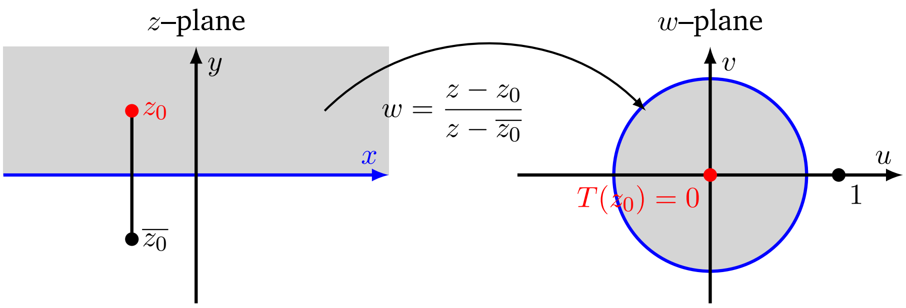
Remark.
The above mappings are not unique, because if $T(z)$ is a bilinear map between a half plane and the unit disc then for every $\alpha \in \mathbb{R}$ the bilinear mapping $e^{i\alpha}T(z)$ would also map this half plane one-to-one and onto the unit disc.

Problem. Characterize all bilinear transformations that map the upper half-plane $\operatorname{Im} z > 0$

onto the unit disc $|w| < 1$.

Answer. Let $z_0 \in \mathbb{C}$ such that $\operatorname{Im} z_0 > 0$. Define the bilinear transformation

$$w = T(z) = \frac{z - z_0}{z - \overline{z_0}}.$$



Then T has the following characteristics:

1. $T(z_0) = 0$ and $T(\overline{z_0}) = \infty$.
2. T maps the the real line $\operatorname{Im} z = 0$ onto the unit circle $|w| = 1$,
3. T maps the upper half lane line $\operatorname{Im} z > 0$ onto the unit disc $|w| < 1$, and
4. T maps the lower half plane $\operatorname{Im} z < 0$ onto $|w| > 1$.

The most general bilinear transformation of the real line $\mathbb{R}$ onto the unit circle $|w| = 1$, is given by

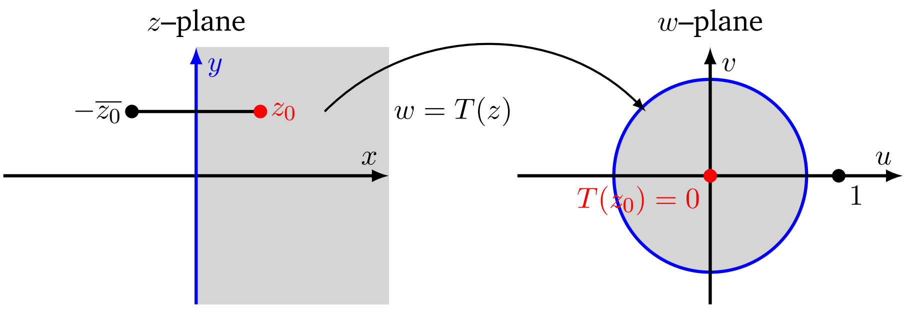
$$w = S(z) = e^{i\alpha} T(z) = e^{i\alpha} \frac{z - z_0}{z - \overline{z_0}}$$

where $\alpha \in \mathbb{R}$ and $\operatorname{Im} z_0 > 0$. Since $S(z_0) = 0$ then S maps the upper half plane onto the unit disc.

Problem. Characterize all bilinear transformations that map the right half plane $\operatorname{Re} z > 0$ onto the unit disc $|w| < 1$.

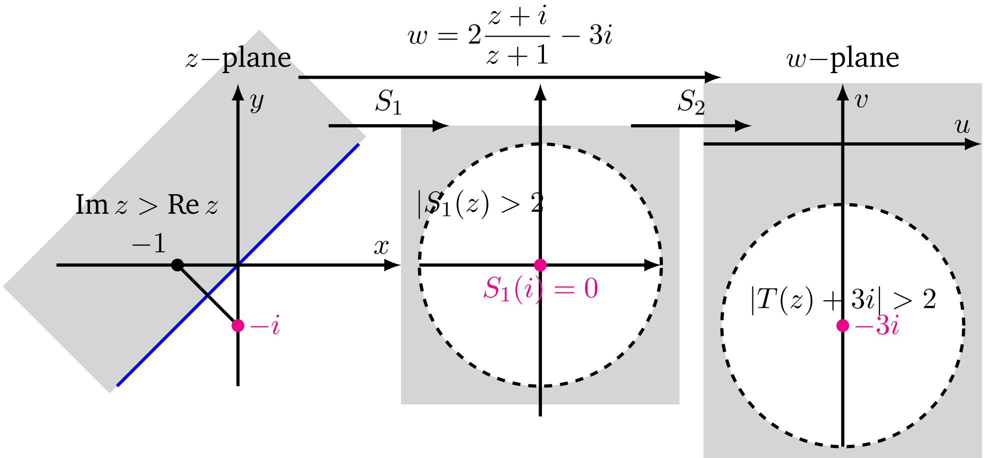
Answer. Let $z_0 \in \mathbb{C}$ such that $\operatorname{Re} z_0 > 0$, $\alpha \in \mathbb{R}$ and define the bilinear transformation

$$w = T(z) = e^{i\alpha} \frac{z - z_0}{z + \overline{z_0}}.$$



Problem. Find $T(A)$ if $A = \{z : \operatorname{Im} z > \operatorname{Re} z\}$ and $T(z) = 2\frac{z + i}{z + 1} - 3i$.

Solution. Note that we can rewrite T as a composition of two simpler transformations as follows: $T(z) = (S_3 \circ S_2)$ where $S_1(z) = 2\dfrac{z+i}{z+1}$ and $S_2(z) = 2z$. Recall that $\dfrac{z+i}{z+1}$ is of the form $\dfrac{z-a}{z-b}$ which transforms the line bissecting the segment joining a and b onto the unit circle $|w| = 1$. In our case $a = -i$ and $b = -1$, and note that the image of $-i$ is the origin. Hence the image $\operatorname{Im} z > \operatorname{Re} z$ is the unit disc $\mathbb{D}$. Henceforth S_1 maps A onto $B = \{w : |w| > 2\}$. $S_2(z)$ translates B by $-3i$ to get $C = \{w : |w+2i| > 2\}$.



4.6 Conformal Self-Maps of the Disc

We call $T : \mathbb{D} \to \mathbb{D}$ a conformal self-map of the unit disc $\mathbb{D}$ or an automorphism of the disc $\mathbb{D}$, if it is conformal and maps the unit disc $\mathbb{D}$ to itself, i.e. $T(\mathbb{D}) = \mathbb{D}$. In this section characterize all conformal maps of the unit disc to itself.

The automorphisms (that is, conformal self-mappings)

For $a \in \mathbb{C}, |a| < 1$, we define

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

Then each φ_a is a conformal self-map of the unit disc.

If $z \in \partial\mathbb{D}$ then $|z| = 1$, and

$$|\varphi_a(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| = \left|\frac{\bar{z}(z-a)}{1-\bar{a}z}\right| = \left|\frac{1-\bar{z}a}{1-\bar{a}z}\right| = \left|\frac{\overline{1-\bar{a}z}}{1-\bar{a}z}\right| = 1.$$

Clearly φ_a maps the unit circle $|z| = 1$ onto itself.

Furthermore since $\varphi_a(a) = 0, a \in \mathbb{D}$, then the $\mathbb{D}$ is mapped onto $\mathbb{D}$. The same reasoning applies to

$$(\varphi_a)^{-1} = \varphi_{-a},$$

hence φ_a is a one-to-one conformal map of the the unit disc $\mathbb{D}$ to itself.

The general form of biholomorphic conformal self maps of the disc is then

$$T(z) = e^{i\alpha}\varphi_a(z) = e^{i\alpha}\frac{z-a}{1-\bar{a}z},$$

where $\alpha \in \mathbb{R}$.
In other words, any conformal self-map of the unit disc to itself is the composition of a Mobius transformation with a rotation. It can also be shown that any conformal self-map of the unit disc can be written in the form

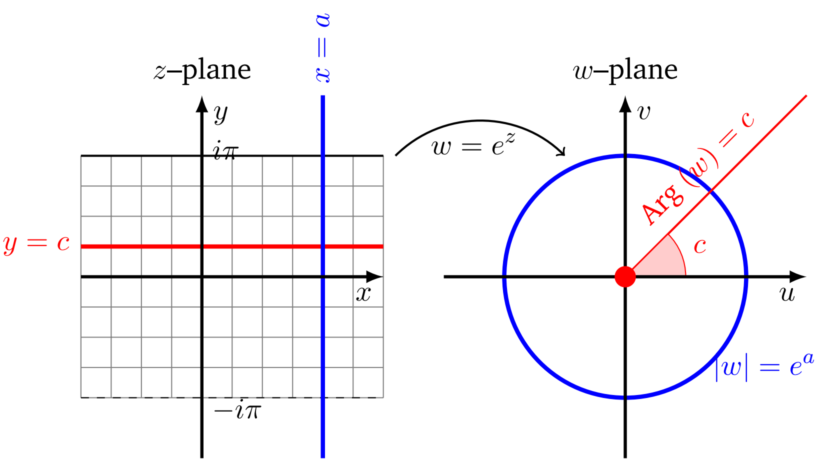
$$T(z) = \varphi_a(e^{i\alpha}z),$$

for some Mobius transformation φ_a and some some real number α.
A special case of these maps are the self maps of $\mathbb{D}$ which fix the origin i.e. $\varphi_a(0) = 0$. Clearly in this case we have $\varphi_a(0) = -a = 0$ and $\varphi_0(z) = z$, hence the general form of such maps is given by

$$T_\alpha(z) = e^{i\alpha}z.$$

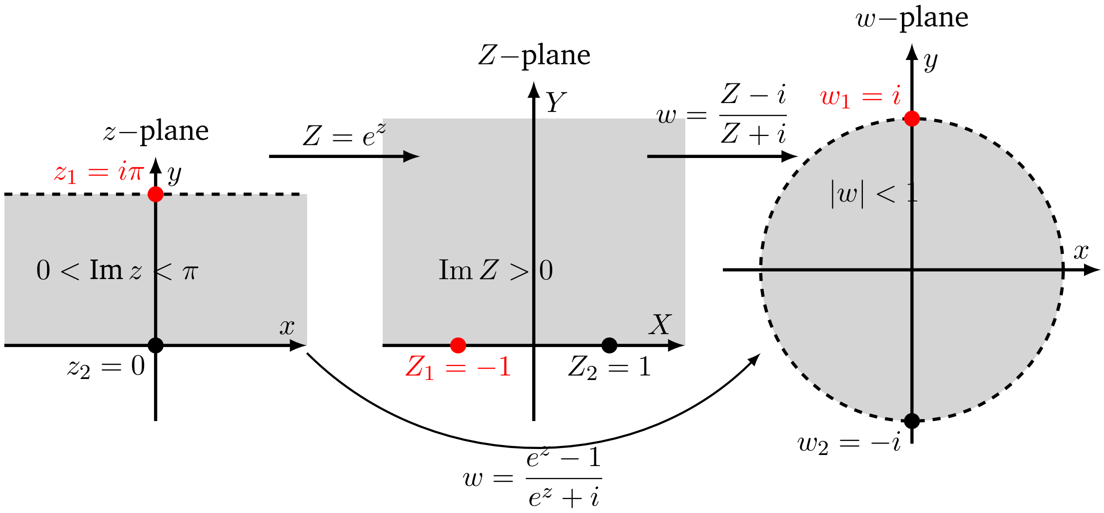
4.7 Compositions of Conformal Transformations

Recall that the function $w = f(z) = e^z$ is a one-to-one mapping of the fundamental period strip $-\pi < y \leq \pi$ in the z plane onto the w plane with the point $w = 0$ deleted. Since $f'(z) \neq 0$, the mapping $w = \exp z$ is a conformal mapping at each point z in the complex plane. The family of horizontal lines $y = c, -\pi < c \leq \pi$ and the segments $x = a$ and $-\pi < y < \pi$ form an orthogonal grid in the fundamental period strip. Their images under the mapping $w = e^z$ are the rays $c > 0$ and $\arg w = c$ and the circles $|w| = e^a$, respectively. These images form an orthogonal curvilinear grid in the w plane, as shown in Figure. The inverse mapping is the principal branch of the logarithm $z = \operatorname{Log} w$.



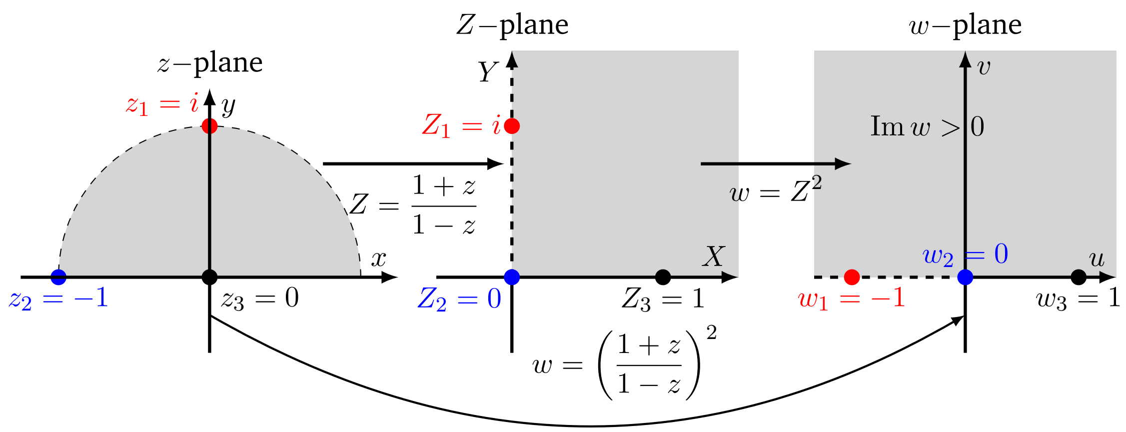
Example 4.9. Show that the transformation $w = f(z) = \dfrac{e^z - i}{e^z + i}$ is a one-to-one conformal mapping of the horizontal strip $0 < y < \pi$ onto the disc $|w| < 1$. Furthermore, the x axis is mapped onto the lower semicircle bounding the disc, and the line $y = \pi$ is mapped onto the upper semicircle.

Solution. Solution The function $w = f(z)$ can be considered as a composition of the exponential mapping $Z = e^z$ followed by the fraction linear transformation $w = \dfrac{Z-i}{Z+i}$. The image of the horizontal strip $0 < y < \pi$ under the mapping $Z = e^z$ is the upper half plane $\mathrm{Im}(Z) > 0$; the x axis is mapped onto the positive X axis; and the line $y = \pi$ is mapped onto the negative X axis. The bilinear transformation $w = \dfrac{Z-i}{Z+i}$ then maps the upper half plane $\mathrm{Im}(Z) > 0$ onto the disc $|w| < 1$; the positive X axis is mapped onto the lower semicircle; and the negative X axis onto the upper semicircle. The figure illustrates the composite mapping.



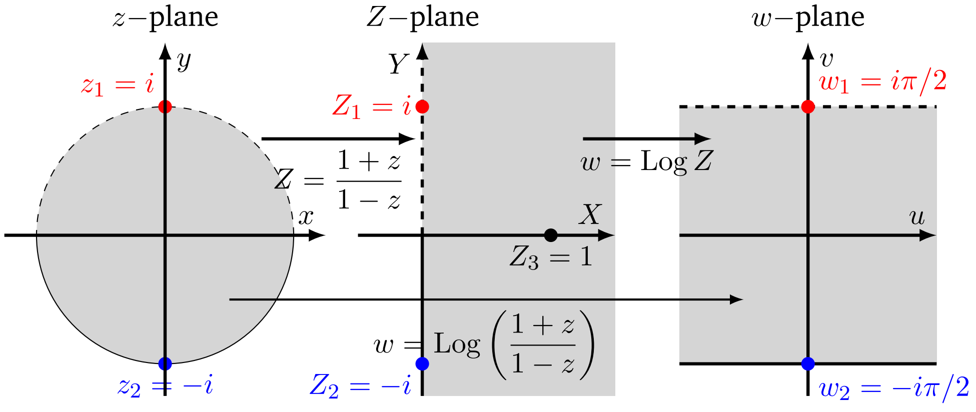
Example 4.10. The transformation $w = f(z) = \left(\dfrac{1+z}{1-z}\right)^2$ is a one- to-one conformal mapping of the portion of the disc $|z| < 1$ that lies in the upper half plane $\mathrm{Im}(z) > 0$ onto the upper half plane $\mathrm{Im}(w) > 0$. Furthermore, the image of the semicircular portion of the boundary is mapped onto the negative u axis, and the segment $-1 < x < 1, y = 0$ is mapped onto the positive u axis.

Solution. The function $w = f(z)$ is the composition of the bilinear transformation $Z = (1+z)/(1-z)$ followed by the mapping $w = Z^2$. The image of the half-disc under the bilinear mapping $Z = (1+z)/(1-z)$ is the first quadrant $X > 0, Y > 0$; the image of the segment $y = 0, -1 < x < 1$, is the positive X axis; and the image of the semicircle is the positive Y axis. The mapping $w = Z^2$ then maps the first quadrant in the Z plane onto the upper half plane $\mathrm{Im}(w) > 0$, as shown in the figure.



Example 4.11. Show that the transformation $w = f(z) = \operatorname{Log}\left(\dfrac{1+z}{1-z}\right)$ is a one-to-one conformal mapping of the unit disc $|z| < 1$ onto the horizontal strip $|v| < \pi/2$. Furthermore, the upper semicircular is mapped onto the line $v = \pi/2$ and the lower semicircular is mapped onto the line $v = -\pi/2$.

Solution. The function $w = f(z)$ is the composition of the bilinear transformation $Z = (1+z)/(1-z)$ followed by the mapping $w = \operatorname{Log} Z$. The image of the disc $|z| < 1$ under the bilinear mapping $Z = (1+z)/(1-z)$ is the right half plane $\operatorname{Re} Z > 0$; the upper semicircle is mapped onto the positive Y axis; and the lower semicircle is mapped onto the negative Y axis. The logarithmic function $w = \operatorname{Log} Z$ then maps the right half plane onto the horizontal strip; the image of the positive Y axis is the line $v = \pi/2$; and the image of the negative Y axis is the line $v = -\pi/2$.



Example 4.12. Show that the bilinear mapping

$$w = T(z) = \frac{(1-i)z + 2}{(1+i)z + 2}$$

maps the disk $|z+1| < 1$ onto the upper half plane $\operatorname{Im}(w) > 0$.

Solution. Fist we show that T maps the circle $|z+1| = 1$ onto the real line $\operatorname{Im}(w) = 0$. The map T has a pole at $z = -1+i$ which belongs to the circle , hence $T(-1+i) = \infty$. Furthermore we have $T(-1-i) = 0$ and $T(0) = 1$. That is we have $T\{-1-i, 0, -1+i\} =$

$\{0, 1, \infty\}$ and the three points on the circle of the z plane are mapped on u axis of the w plane, hence the circle $|z+1| = 1$ is mapped onto the u axis. Since $T(-1) = i$ then maps the disk $|z+1| < 1$ onto the upper half plane $\mathrm{Im}(w) > 0$.

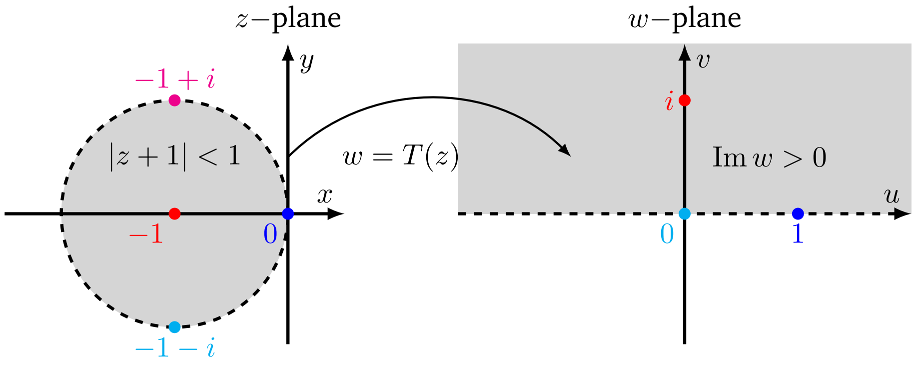


Figure 4.13: Image of $|z+1| < 1$ under the map $w = T(z) = \dfrac{(1-i)z+2}{(1+i)z+2}$.

Problem. Find a transformation that maps the set $A = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0, \mathrm{Re}\, z > 0\}$ onto the unit disk $\mathbb{D}$?